Final exam

Lecture spring 2021: General Relativity (FYS4160)

$\rightsquigarrow$ *Carefully* ***read the whole question*** *before you start to answer it! Note that you do not have to answer the problems in the order presented here, so try to answer those first that you feel most sure about. Keep your descriptions as short and concise as possible!*

Remember that you can use 'everything' that you can find in the lecture notes, textbooks or online – but reproducing any text or mathematical derivation literally will be considered cheating (unless properly marked as citation). You also must state your answer based on what was actually covered in this course (as found in the lecture notes and exercises); in particular, there should be no need to use any new notation, results or conventions whatsoever – but if you do, you must first introduce this properly, in your own words, by explicitly connecting it to the corresponding results and contents found in the lecture notes or exercises.

Answers given in English are preferred – but feel free to write in Scandinavian if you struggle with formulations! This exam consists of 5 problems (and subproblems), distributed on 4 pages. Maximal number of available points: 45.

Good luck !!!

<u>Problem 1</u> : Principle of Equivalence *(10 points)*

a) A slinky is hanging stretched under its own weight, as shown in the figure. When released, the top end will accelerate downwards and the bottom end will remain at rest, apparently freely floating in the air, until the whole slinky has collapsed; from that point on, the slinky will fall in a fully collapsed state. Argue why this is a demonstration of the equivalence principle, i.e. why any other behaviour would violate this principle! *(4 points)*

b) State the equation of motion of a test particle in special and general relativity, in arbitrary coordinates, and discuss the difference! *(3 points)*

c) In the lecture we used the result from thermodynamics that the phase space distribution of a 'plasma' of particles in thermodynamic equilibrium is given by

$$f(\mathbf{p}) = \frac{1}{e^{(E-\mu)/T} \pm 1}\,, \tag{1}$$

where T denotes the temperature, μ is the chemical potential and E the energy of the particle, and the plus (minus) sign applies to fermions (bosons). This expression must depends on the frame / coordinate system – why? Bring it into a manifestly covariant form, by writing down the phase-space density measured by an observer moving with 4-velocity v^μ, in an arbitrary geometry! *(4 points)* *[Hint: You need to introduce another 4-velocity for that task; what does that 4-velocity describe physically?]*

<u>Problem 2</u> : Coordinate transformations in Schwarzschild *(10 points)*

The Schwarzschild metric in Schwarzschild coordinates (t, r) takes the standard form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \quad f(r) = 1 - \frac{2GM}{r}. \tag{2}$$

a) Show that the metric

$$ds^2 = -f(r)\, dT^2 + 2C(r)\, dT dr + \frac{1 - C(r)^2}{f(r)} dr^2 + r^2 d\Omega^2 \tag{3}$$

is equivalent to the Schwarzschild metric for any function $C(r)$. *(2 points)* *[Hint: Begin with (2) and consider the coordinate transformation $T(t, r) = t + \psi(r)$, for arbitrary $\psi(r)$.]*

b) Now choose $C(r)$ such that $g_{rr} = 1$. Show that this metric is non-singular for all $r > 0$ (in particular for $r \to 2GM$), i.e. show that all metric components are finite and the metric determinant is non-zero (apart from coordinate singularities on the 2-sphere; why is it impossible to avoid the latter, at least in one point?). *(5 points)*

c) For which choice of $C(r)$ do we recover the Eddington-Finkelstein coordinates (v, r) introduced in the lecture? Demonstrate explicitly that $t+\psi$, as introduced in a), in that case indeed becomes $T = t + r^*$, with the tortoise coordinate $r^* = r + 2GM \log[r/(2GM) - 1]$. *(3 points)*

<u>Problem 3</u> : Stress-energy tensor in Einstein-Maxwell theory *(12 points)*

a) In general relativity the stress-energy tensor associated to an arbitrary Lagrangian $\mathcal{L}_\mathrm{m}$ describing matter (fields) is given by

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\mathcal{L}_\mathrm{m}\right)}{\partial g^{\mu\nu}}, \tag{4}$$

where g denotes the determinant of the metric tensor $g^{\mu\nu}$. Using the field equations, and results obtained in the lecture, show that this definition implies *(3 points)*

$$\nabla^\mu T_{\mu\nu} = 0\,. \tag{5}$$

b) Briefly explain the physical significance of the property of $T_{\mu\nu}$ given in Eq. (5). In particular, what is the main qualitative difference to the analogous property in flat spacetime? *(3 points)*

c) Now let us look at a concrete example, the Maxwell action in a gravitational background:

$$S\left[A_\alpha, g_{\alpha\beta}\right] = -\frac{1}{4} \int d^4x\, \sqrt{-g} F_{\alpha\beta} F^{\alpha\beta}\,, \tag{6}$$

where $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ and A^α is the 4-potential, incorporating both the electric and magnetic potentials, $A^\mu = (\Phi, \vec{A})$. Using this action, show that the energy-momentum tensor of an electromagnetic field in a gravitational background is given by *(3 points)*

$$T_{\alpha\beta} = F_{\alpha\gamma}F^\gamma_\beta - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}. \tag{7}$$

[Hint: You may recall a useful result for $\partial\sqrt{-g}/\partial g^{\mu\nu}$ directly from the lecture...]

d) The equations of motion

$$\nabla_\alpha F^{\alpha\beta} = 0 \quad , \quad \nabla_\alpha F_{\beta\gamma} + \text{ cyclic permutations } = 0 \tag{8}$$

follow from varying Eq. (6) with respect to the fields A^μ. Use this form of Maxwell's equations to explicitly confirm that $\nabla^\alpha T_{\alpha\beta} = 0$ in Maxwell-Einstein theory. *(3 points)*
[Hint: for such calculations you should use the properties of the covariant derivative. If done correctly, this should be a four-line calculation.]

Problem 4 : Gravitational waves from a nearby source *(8 points)*

Imagine that the Chinese space station will be ready at some point, at a size similar to that of the ISS. Unfortunately there is a bug in the code keeping it in orbit, which leads to a head-on collision of the two space stations. Calculate the amplitude of the gravitational waves produced and, based on the result, estimate whether they could be observed on the surface of the earth! Would the conclusion change for *i)* a faster deceleration during the impact (see below for numerical values) or *ii)* a future space-based interferometer, by chance being located closer to the collision point?
[The ISS weighs approximately 400 tons, and orbits Earth with a speed of roughly 30 000 km/h at a (varying) altitude of around 400 km. Assume that the same goes for the Chinese station, and that the two stations decelerate at a constant rate during the collision, coming to rest in about one millisecond. You may treat the space stations as point masses, and neglect the effect of Earth's gravitational field.]

Problem 5 : Density perturbations in the Early Universe *(5 points)*

Sketch the evolution of density perturbations as a function of time (conveniently expressed in terms of the scale factor a), for the dark matter, radiation and baryon component, respectively (all in the same figure). For the spatial scale, assume very roughly the size of a galaxy cluster (corresponding to the largest bound objects), and that this scale is well inside the horizon for all values of a that you plot. Also indicate in the figure the evolution of the baryon overdensities without the presence of a dark matter component. Argue why these two 'baryon lines', in comparison, provide strong evidence for the fact that the total matter component in the Universe must be dominated by dark matter.

Bonus questions (3 more points): How would the plot look like for very early times (but still after inflation), when the scale of the perturbation for which you plotted the evolution is no longer much smaller than the horizon? Imagine that you had drawn the evolution of density perturbations for different scales – would the baryon or the dark matter line depend more strongly on when the corresponding perturbation entered the horizon (inside the horizon, but before CMB times)? Based on that, what is (roughly) the size of density perturbations in the baryon component that you expect today, for scales larger than the horizon scale at recombination?