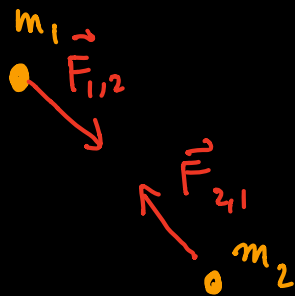


5.2. Schwarzschild geodesics

Reminder: The Kepler problem



$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 \quad ; \quad \hat{r} \equiv \frac{\vec{r}}{|\vec{r}|}$$

$$\vec{F}_{1,2} = -G \frac{m_1 m_2}{r^2} \hat{r} = -\vec{F}_{2,1}$$

Newton's 2nd law:
$$\left. \begin{aligned} \vec{F}_{1,2} &= m_1 \frac{d^2 \vec{r}_1}{dt^2} \\ \vec{F}_{2,1} &= m_2 \frac{d^2 \vec{r}_2}{dt^2} \end{aligned} \right\} \frac{d^2 \vec{r}}{dt^2} = \frac{\vec{F}_{1,2}}{m_1} - \frac{\vec{F}_{2,1}}{m_2} \quad \text{Newton's 3rd law}$$
$$= \vec{F}_{1,2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right)$$
$$\equiv \frac{1}{\mu}$$

$\leadsto$ one body problem with "reduced mass" $\mu \simeq m_1$ for $m_1 \ll m_2$

$$\mu \frac{d^2 r^2}{dt^2} = -G \frac{m_1 m_2}{r^2} \hat{r}$$

constants of motion: • energy E ($\Leftarrow$ no external force)

• angular momentum $\vec{L}$ ($\Leftarrow$ no torque
 $\Leftarrow$ radial force)

$\Rightarrow$ orbit lies in
plane $\perp \vec{L}$

$$\Rightarrow E = T + U$$

$$= \frac{1}{2} \mu \left(\frac{d\vec{r}}{dt} \right)^2 - G \frac{m_1 m_2}{r}$$

$$= \frac{1}{2} \mu \left[\left(\frac{dr}{dt} \right)^2 + \left(r \frac{d\phi}{dt} \right)^2 \right] - G \frac{m_1 m_2}{r}$$

$\vec{r} = r \cdot \hat{r}$
 $\hat{r} = \hat{r}(\phi, \theta)$
 $= \hat{r}(\phi)$
 $\frac{d\vec{r}}{dt} = \frac{dr}{dt} \hat{r} + r \frac{d\hat{r}}{dt}$
 $= \frac{dr}{dt} \hat{r} + r \left(\frac{\partial \hat{r}}{\partial \phi} \frac{d\phi}{dt} \right)$
 $\quad \quad \quad \hat{\phi}$

$$| L = |\vec{L}| = \mu r v_{tan}$$

$$= \mu r^2 \frac{d\phi}{dt}$$

$$\Rightarrow r \frac{d\phi}{dt} = \frac{L}{\mu r}$$

$$\Rightarrow \boxed{E = \frac{1}{2} \mu \left(\frac{dr}{dt} \right)^2 + \frac{1}{2} \frac{L^2}{\mu r^2} - G \frac{m_1 m_2}{r}}$$

$\underbrace{\hspace{10em}}_{\equiv U_{eff}(r)}$

$\Rightarrow$ reduced the system to a single d.o.f. !
(1 particle in 1D)

$$\Rightarrow \frac{dr}{dt} = \sqrt{\frac{2}{\mu} (E - U_{eff}(r))}$$

$\Rightarrow$ a) integrate $\leadsto r(t)$

b) $\frac{d\phi}{dr} = \frac{d\phi}{dt} \cdot \left(\frac{dr}{dt} \right)^{-1} \leadsto$

$$\boxed{r = \frac{r_0}{1 - e \cos \phi}}$$

orbit equation

eccentricity

μr^2

$\leadsto$ 4 cases: $E > 0$ ($\Rightarrow \epsilon > 1$): hyperbola

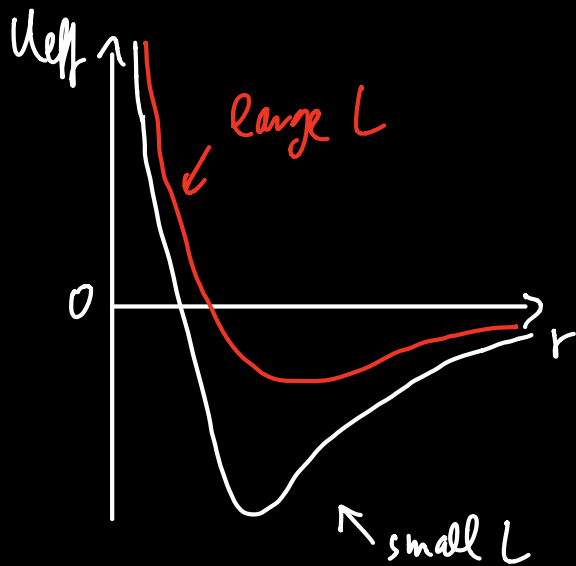
$E = 0$ ($\Rightarrow \epsilon = 1$): parabola

$E_{\min} < E < 0$ ($\Rightarrow 0 < \epsilon < 1$): ellipse

$E_{\min} = E$ ($\epsilon = 0$): circle

$$\uparrow E_{\min} = - \frac{\mu G^2 m_1^2 m_2^2}{2L^2}$$

qualitative picture already directly from $U_{\text{eff}}(r)$:



$\Rightarrow$ e.g.

• circular orbits ($r = r_c = \text{const.}$) at

$$U'_{\text{eff}}(r_c) = 0$$

$$-\frac{L^2}{\mu r_c^3} + \frac{G m_1 m_2}{r_c^2} = 0$$

$$\Leftrightarrow \boxed{r_c = \frac{L^2}{\mu G m_1 m_2}}$$

$$\Rightarrow E_{\min} = U_{\text{eff}}(r_c)$$

• elliptical orbits:

r oscillates around r_c

• $E > 0 \Rightarrow$ infalling particle turns

at some r_{min} , then
escapes to $r \rightarrow \infty$

effective potential in Schwarzschild geometry

$$x^{\mu} = (t, r, \theta, \phi)$$

$\Rightarrow$ 9 non-vanishing $\Gamma^{\mu}_{\alpha\beta}$ so

$\Rightarrow$ 4 relatively complicated equations for $\frac{dx^{\mu}}{d\lambda^2}$
from geodesic equation

alternative to full solution: understand orbits from
symmetries (c.f. Kepler!)

a) K is Killing vector

$$\Rightarrow K_{\mu} \frac{dx^{\mu}}{d\lambda} = \text{const.}$$

$\uparrow$ p^{μ} for photons / massless particles

$\frac{1}{m} p^{\mu}$ - massive particles

i) 3 Killing vectors for spherical symmetry

$\Rightarrow$ conservation of angular momentum $\vec{L}$

• 2 Killing vectors for direction $\vec{L}/|\vec{L}|$

$\Rightarrow$ particles move in plane

$\Rightarrow$ can choose $\theta = \frac{\pi}{2}$ (equatorial plane)

- 1 Killing for magnitude of $\vec{L}$

$$R^\mu = (\partial_\phi)^\mu = (0, 0, 0, 1)$$

$$\Rightarrow R_\mu = g_{\mu\nu} R^\nu = (0, 0, 0, r^2 \sin^2 \theta) = (0, 0, 0, r^2)$$

$$\Rightarrow L \equiv R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda} \text{ is conserved}$$

$$= \begin{cases} \text{angular momentum per unit mass} & \text{for } m > 0 \\ \text{angular momentum} & m = 0 \end{cases}$$

ii) 1 time-like Killing vector

$$K^\mu = (\partial_t)^\mu = (1, 0, 0, 0)$$

$$\Rightarrow K_\mu = g_{\mu\nu} K^\nu = \left(-\left(1 - \frac{2GM}{r}\right), 0, 0, 0 \right)$$

$$= E \equiv -K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\lambda} \text{ is conserved}$$

$$= \begin{cases} \text{energy per unit mass} & \text{for } m > 0 \\ \text{energy} & \text{for } m = 0 \end{cases} \quad (\text{c.f. } r \rightarrow \text{limit!})$$

b) always for geodesics:

$$- g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \mathcal{E} = \text{const.}$$

- $\lambda = \tau$ (massive particles) $\Rightarrow \mathcal{E} = -g_{\mu\nu} u^\mu u^\nu = +1$
- (massless particles) $\Rightarrow \mathcal{E} = 0$

τ_λ is instead fixed by
 $p^\mu = \frac{dx^\mu}{d\lambda}$

$$\Rightarrow \mathcal{E} = \left(1 - \frac{2GM}{r}\right) \left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2GM}{r}\right)^{-1} \left(\frac{dr}{d\lambda}\right)^2 - r^2 \left(\frac{d\Omega}{d\lambda}\right)^2$$

$$\nearrow = \left(\frac{d\phi}{d\lambda}\right)^2$$

$$\theta = \frac{\pi}{2} = \text{const.}$$

$$\left| \begin{array}{l} \bullet \frac{dt}{d\lambda} = \frac{E}{\left(1 - \frac{2GM}{r}\right)} \\ \bullet \frac{d\phi}{d\lambda} = \frac{L}{r^2} \end{array} \right.$$

$$\bullet \frac{d\phi}{d\lambda} = \frac{L}{r^2}$$

$$\Rightarrow \mathcal{E} = \left(1 - \frac{2GM}{r}\right)^{-1} \left\{ E^2 - \left(\frac{dr}{d\lambda}\right)^2 \right\} - \frac{L^2}{r^2} \quad \left| \times -\frac{1}{2} \left(1 - \frac{2GM}{r}\right) \right.$$

$$(\Rightarrow) \underbrace{\frac{1}{2} (E^2 - \mathcal{E})}_{= \text{const.}} = + \underbrace{\frac{1}{2} \left(\frac{dr}{d\lambda}\right)^2 - \mathcal{E} \frac{GM}{r} + \frac{L^2}{2r^2}}_{U_{\text{eff}}(r)} - \underbrace{\frac{GM L^2}{r^3}}_{\text{not present in Newtonian}}$$

not present
in Newtonian

$\Rightarrow$ identical behaviour for $r \gg 2GM$!

comments : • the above is an exact equation
(not an expansion in $1/r$)

• for $m > 0 \leadsto E = +1$:

$$E = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau} \stackrel{r \gg 2GM}{\approx} 1 + \underbrace{\frac{E_{ki}}{m}}_{\ll 1 \text{ in Newtonian limit}}$$

$$= \frac{m + E_{ki}}{m}$$

$$\Rightarrow \mathcal{E} \equiv \frac{1}{2} (E^2 - 1) = \frac{1}{2} \left(2 \frac{E_{ki}}{m} + \frac{E_{ki}^2}{m^2} \right) \approx \frac{E_{ki}}{m}$$

$\leadsto$ same as l.h.s. in Newtonian expression

$\Rightarrow \mathcal{E}$ is the kinetic energy for $r \gg 2GM$

• at smaller r : interpretation of "energy" changes, but equation still of the same form

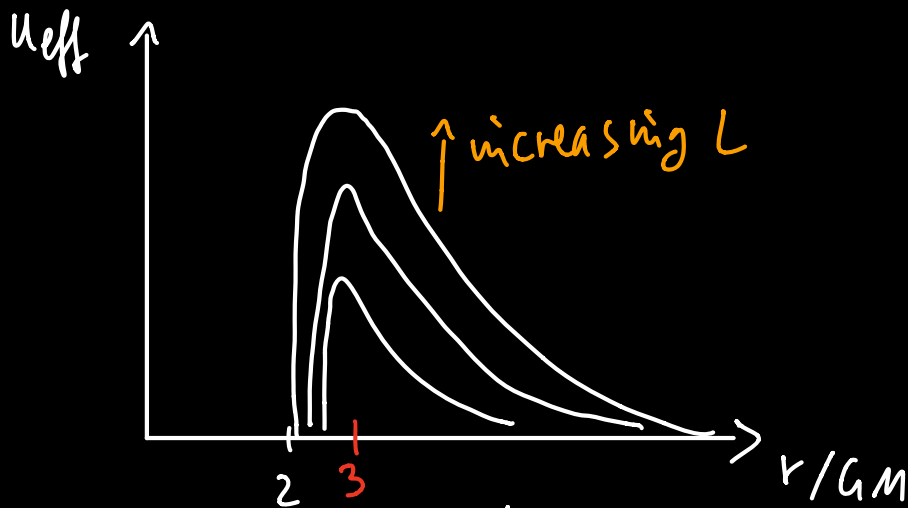
$\Rightarrow$ as before, circular orbits for $U'_{\text{eff}} = 0$

$$\Rightarrow +\epsilon \frac{GM}{r_c^2} - \frac{L^2}{r_c^3} + \frac{3GM L^2}{r_c^4} = 0$$

$$\Rightarrow \epsilon GM r_c^2 - L^2 r_c + 3GM L^2 = 0$$

a) massless particles

$\epsilon = 0 \Rightarrow \boxed{r_c = 3GM} : \text{unstable circular orbits}$



"black
hole"

"Newtonian regime" (NB: $m=0$)

b) massive particles

$$\epsilon = 1 \Rightarrow r_c = \frac{L^2}{2GM} \left(1 \pm \sqrt{1 - 12GM^2/L^2} \right)$$

i) large $L \Rightarrow \sqrt{r} = 1 - 6GM^2/L^2$

$\Rightarrow r_c = \left(3GM, \frac{L^2}{GM} \right)$

↑
unstable

$\equiv m=0$ case

↑
stable orbit

$\equiv$ Newtonian case

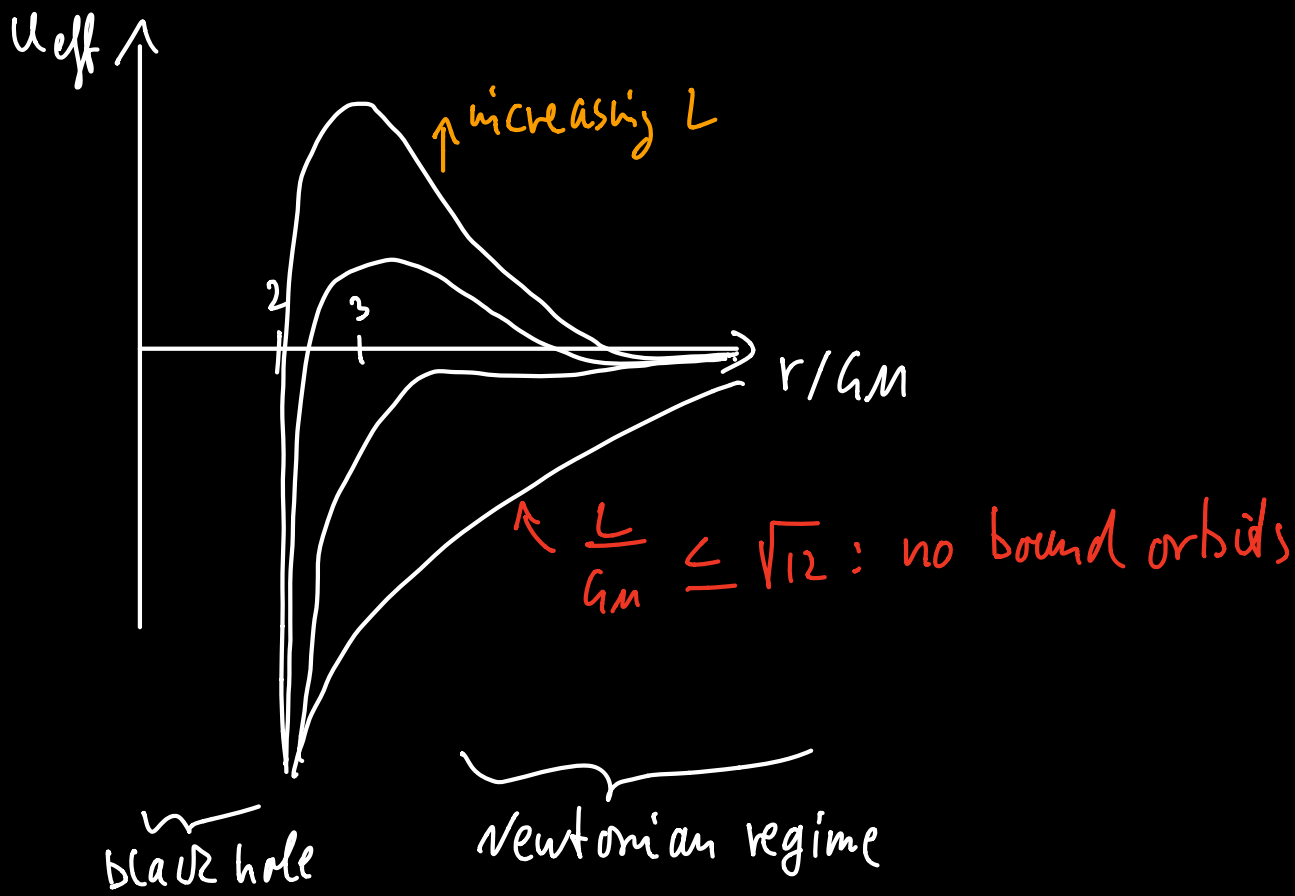
ii) smallest circular orbit: $r = 0$

$\Rightarrow L = \sqrt{12} GM \Rightarrow \boxed{r_c = 6GM}$

• bound, but non-circular orbits oscillate around r_c

$\rightarrow$ Newtonian case: ellipses

$G R = : =$ with perihelion precession
 $\leadsto$ discuss later



- relevant length scales : $2 G M_{\odot} \approx 3 \text{ Km} \ll r_{\odot} = 7 \cdot 10^8 \text{ m}$
 $2 G M_{\oplus} \approx 0.9 \text{ cm} \ll r_{\oplus} = 6 \cdot 10^6 \text{ m}$