

5.3 Schwarzschild black holes

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

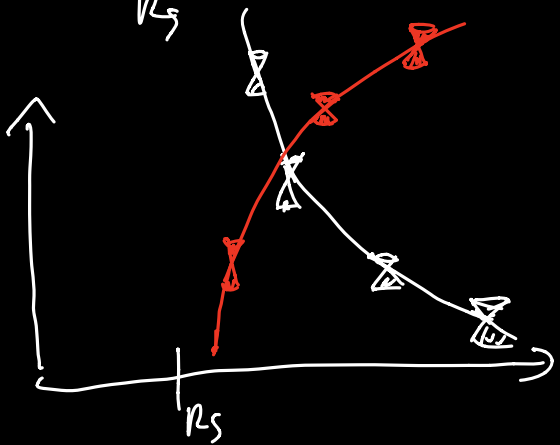
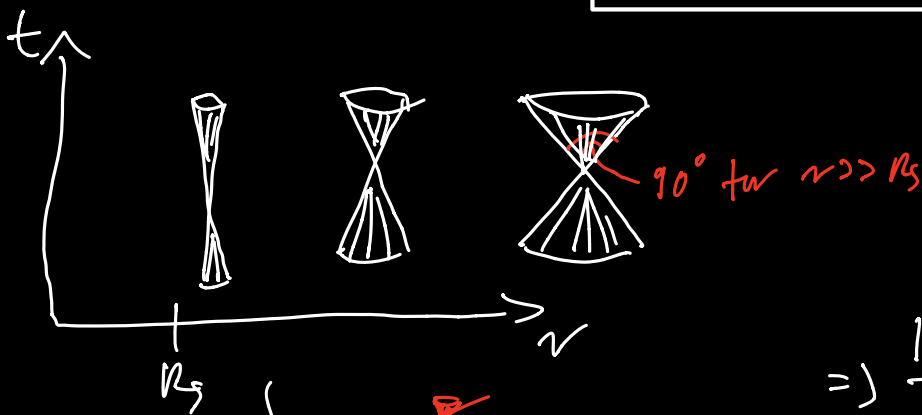
consider radial ($\Rightarrow d\Omega=0$) null curves ($\Rightarrow ds^2=0$):

$$0 = -\left(1 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2$$

$$\Rightarrow \frac{dt}{dr} = \pm \left(1 - \frac{R_S}{r}\right)^{-1} = \begin{cases} \pm 1 & \text{for } r \gg R_S \\ \pm \infty & \text{for } r \rightarrow R_S \end{cases}$$

$$\Rightarrow t = \text{const.} \pm r^*$$

$$r^* \equiv r + R_S \ln\left(\frac{r}{R_S} - 1\right)$$



$\Rightarrow$ In these co-ordinates,
a light ray from $r \gg R_S$
never seems to reach $r = R_S$!

$\Rightarrow$ we can never see an
infalling observer reach
 $r = R_S$!

Q: Do Schwarzschild ^{coordinates} cover the whole manifold?

$\Rightarrow$ idea: choose alternative time-like coordinate:

$$t \rightarrow u \equiv t - r^*$$

$$v \equiv t + r^*$$



"Eddington - Finkelstein
coordinates"

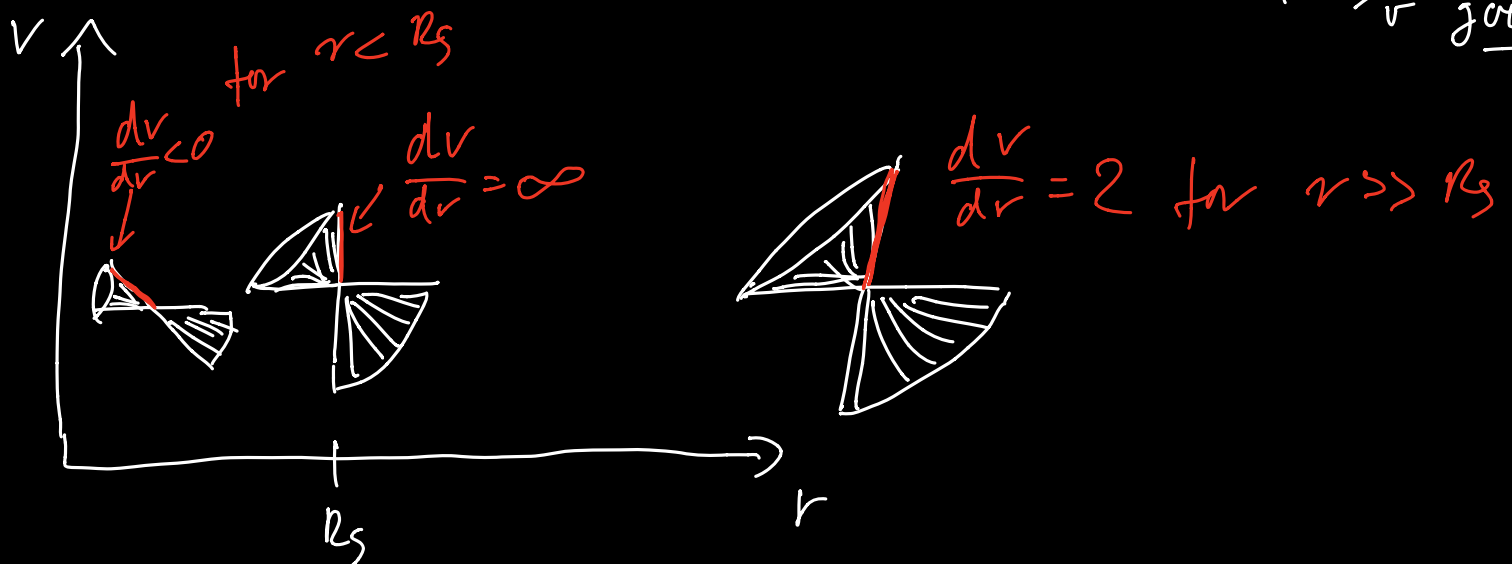
$$\Rightarrow \dots \boxed{ds^2 = -\left(1 - \frac{R_S}{r}\right) dv^2 + 2 dv dr + r^2 d\Omega^2}$$

NB: $g_{rr} = -\left(1 - \frac{R_S}{r}\right) \rightarrow 0$ for $r \rightarrow R_S$

but $g = \underbrace{\det \begin{pmatrix} -\left(1 - \frac{R_S}{r}\right) & 1 \\ 1 & 0 \end{pmatrix}}_{-1} \cdot r^4 \sin^2 \theta \neq 0$ for $r = R_S$!

radial null curves:

$$0 = -\left(1 - \frac{R_S}{r}\right) \left(\frac{dv}{dr}\right)^2 + 2 \left(\frac{dv}{dr}\right) \Rightarrow \frac{dv}{dr} = \begin{cases} 0 & : \text{ingoing} \\ \frac{2}{1 - \frac{R_S}{r}} & : \text{outgoing} \end{cases}$$



$\Rightarrow$ conclusions: a) "new" part of manifold discovered by following future-directed null geodesics

NB: conclusion is - b) for $r < R_S = 2GM$, all future-directed paths are in the direction of decreasing r !

choice because $ds^2 = \text{time-like/spacelike}$ is independent of coordinate choice $\Rightarrow$ impossible to see inside R_S : "black hole"

technical def. of "event horizon"

$\Rightarrow$ surface past which nothing
can escape to infinity

NB: global
concept!

Q: Are there further regions of the full manifold?

A: yes, can be reached by following

i) past-directed geodesics [choose $v \rightarrow u$]

ii) space-like geodesics

Q: $\exists$ global coordinate system to describe the
entire manifold?

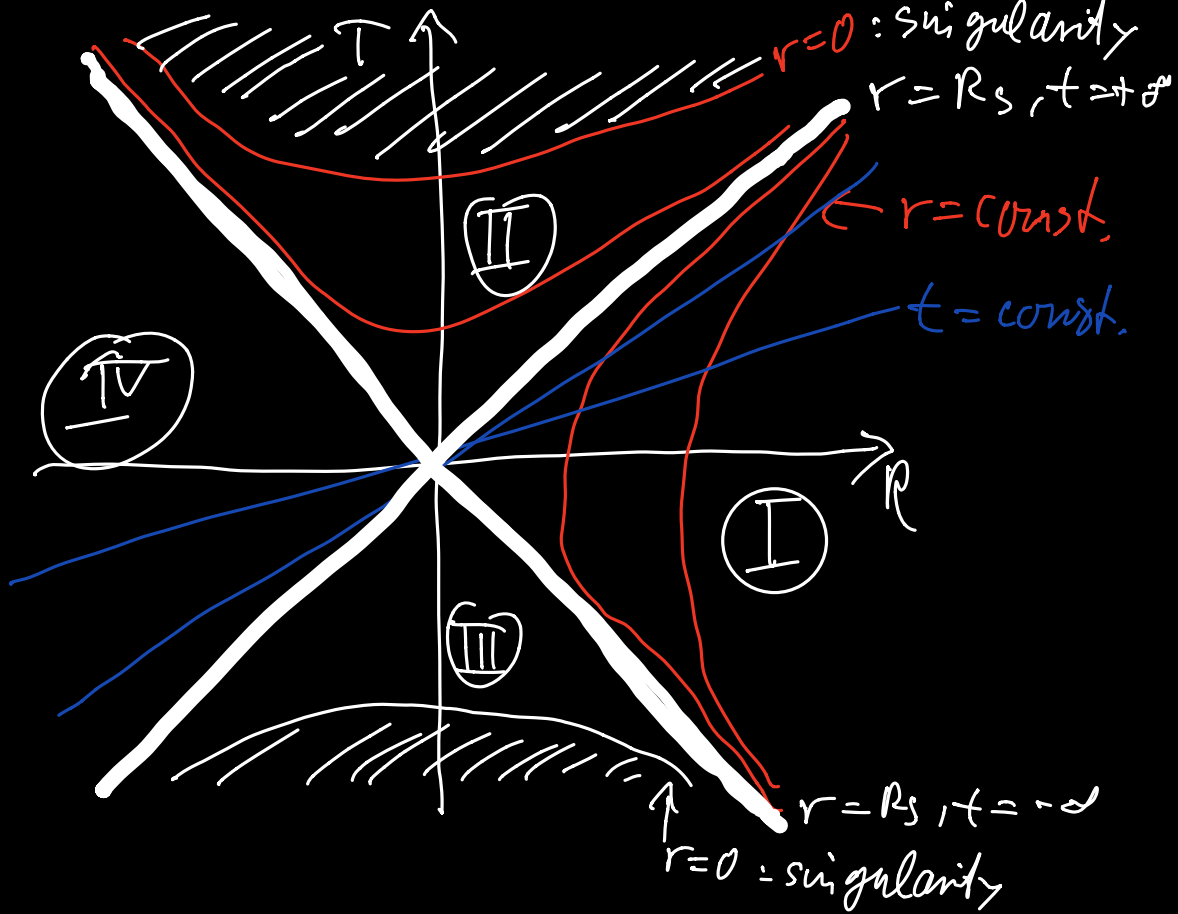
A: yes, Kruskal coordinates

$$T, R \equiv \frac{1}{2} \left(e^{\frac{v}{2R_s}} \mp e^{-\frac{u}{2R_s}} \right)$$

$$\Rightarrow \boxed{ds^2 = \frac{4R_s^2}{r} e^{-\frac{r}{R_s}} (-dT^2 + dR^2) + r^2 d\Omega^2}$$

where $r = r(T, R)$ defined by $T^2 - R^2 \equiv \left(1 - \frac{r}{R_s}\right) e^{\frac{r}{R_s}}$

NB: $ds^2 = 0 \Rightarrow \frac{dT}{dR} = \pm 1$ = (radial) light cones
& $dR=0$ everywhere at $\pm 45^\circ$!



- (I) : $r > R_s$: "our universe"
- (II) : black hole : every future-directed path will hit the singularity at $r=0$
- (III) : "white hole" : bounded by "past event horizon"
 → everything in (I) seems to spring from past singularity
- (IV) : "mirror universe", causally disconnected from (I)

Conformal / "Penrose" diagrams

→ powerful way of representing both global properties and causal structure of (sufficiently symmetric) spacetimes

main idea: • choose coordinates where lightcones have $\pm 45^\circ$

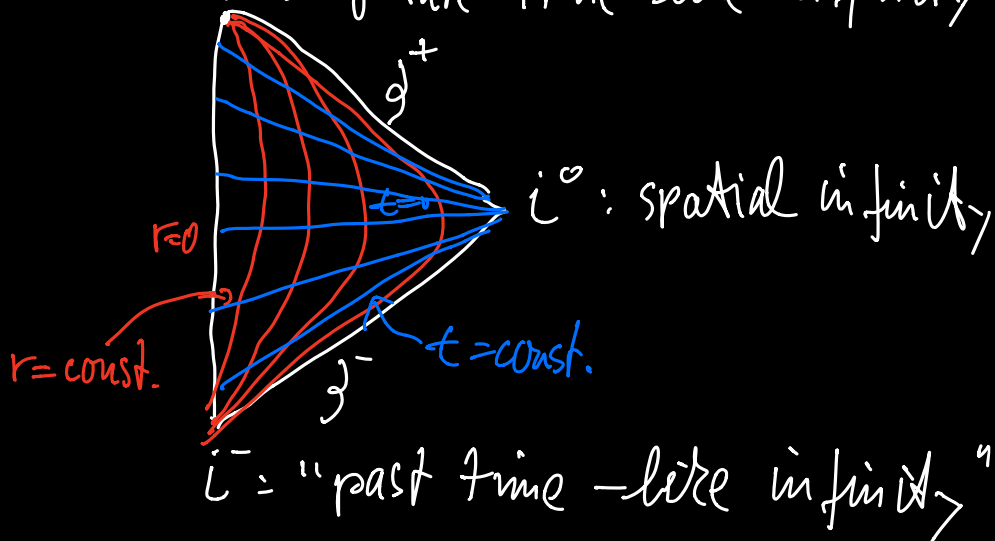
• perform a "conformal transformation":

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} \equiv \omega^2(x) g_{\mu\nu}$$

→ light cones are unaffected ($ds^2=0$)

• choose $\omega(x)$ such that entire manifold is described by a finite range of coordinates.

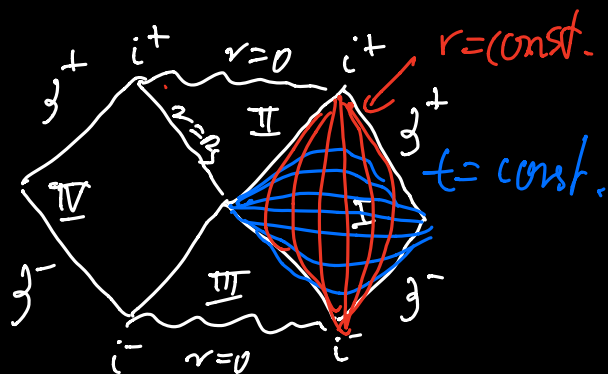
e.g. a) all of Minkowski (every point is a 2-sphere)
 i^+ : "future time-like infinity"



$\mathcal{I}^\pm$: future / past null infinity

$\Rightarrow$ all time-like geodesics begin/end at i^- / i^+
= null = = = $\mathcal{I}^- / \mathcal{I}^+$

b) Schwarzschild spacetime:



black hole: singularity hidden behind the event horizon

white hole : "naked" singularity, $\Rightarrow$ not hidden behind an event horizon

"cosmic censorship conjecture"

~ naked singularities cannot form in gravitational collapse from a "generic", asymptotically flat spacetime.