

Solutions [and grading guidelines] to final exam in FYS4160

[General grading guidelines: Stated points are given for arriving at the respective expression in a fully satisfactory way. Whole point subtractions for bad logic, physically wrong statements or expressions that are *physically* wrong (e.g. wrong dimensions)! ‘Obvious’ small math errors (e.g. wrong prefactors) only 0.5 pt. When possible, no subtraction for follow-up mistakes. Up to 1 extra point for outstanding explanations or demonstrating special insight (but this cannot result in more points than available for a given problem).]

Problem 1

a) The generalization of Newton’s second law in special relativity (SR) is given by the geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = f^\mu, \quad (1)$$

where τ is the eigentime (or an affine parameter) and f^μ a four-vector representing the force [1 pt]. The necessary condition on $\mathbf{F}$ is that such a four vector exists, with the property that $f^i = F^i/m$ (at least) in the non-relativistic limit [1 pt]. This is for example the case of for the Coulomb force – as witnessed by the fact that Maxwell’s equations are Lorentz invariant – but not for Newtonian gravity (for which no consistent relativistic generalization in this sense exists) [0.5 pts each for any valid example]. In GR, formally the same equations govern the dynamics – the difference being that all gravitational effects / forces are already taken into account in the l.h.s. (and hence are not part of f^μ), which in turn implies that in general there is no coordinate system in which $\Gamma_{\rho\sigma}^\mu = 0$ for any space-time *region* (while local coordinates will always make this possible at any given *point*) [1 pt for correctly pointing out the difference.]

b) Following the hint, we know that the surface gravity is given by $a_r = GM/r^2 \approx 1/(4GM)$ just above the horizon at $r = 2GM$, i.e. where the Hawking radiation is produced [1 pt]. According to the equivalence principle, an observer accelerating with constant a_r through vacuum – without any gravitational field involved – must experience exactly the same physics [1 pt]. Hence, the Unruh temperature is given by $T = a/(2\pi)$. [1 pt]

(Note that the spacetime visible to an observer constantly accelerating in Minkowski space is also bound by a horizon, making the analogy and hence use of the equivalence principle (almost) complete. To see this, consider the situation in 1D. Then, $d^2x/d\tau^2 = a = \text{const.}$ implies $x = \frac{1}{2}a\tau^2 + u_0\tau + x_0$, with u_0 and x_0 being constants and we assume without loss of generality $a > 0$, i.e. acceleration in the positive x -direction. A light ray also travelling in the positive x -direction satisfies $x_{\text{null}} = \tau + y_0$ and will still forever be hidden to the observer if $x_{\text{null}} < x$ for all τ – i.e. for $y_0 < x_{\text{min}} \leq \frac{1}{2}a\tau^2 - (1 - u_0)\tau + x_0$, where x_{min} is the minimum of the quadratic expression in τ on the r.h.s.. [0.5 bonus point for noting a potential ‘difference’ between the two situations, due to the presence of a horizon, another 0.5 for realizing that it’s no difference after all.])

c) As per definition – see one of our derivations of the geodesic equation – the trajectory that maximizes the eigentime between any two space-time points is free fall. Given that we have a very powerful engine at our disposal, we should thus as fast as possible accelerate to a speed close to Earth's escape velocity, remain in free fall until we almost reach the moon, and then at the last moment decelerate abruptly in order to avoid a crash landing. This will maximize the eigentime you experience on board of the space-ship – but otherwise not be very healthy in view of the g -forces involved... **[1 pt for the answer + 1 pt for any reasonable discussion]**

Problem 2

a) Varying the Lagrangian, we obtain as usual the Euler-Lagrange equations **[0.5 pt]**:

$$0 = \delta L = \frac{\partial L}{\partial x^\mu} \delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu} \delta \dot{x}^\mu = \frac{\partial L}{\partial x^\mu} \delta x^\mu - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) \delta x^\mu \rightsquigarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} = 0. \quad (2)$$

With the given form of L , we get

$$\frac{\partial L}{\partial x^\mu} = \frac{1}{2} m g_{\rho\sigma, \mu} \dot{x}^\rho \dot{x}^\sigma \quad \mathbf{[0.5 pt]} \quad (3)$$

$$\frac{\partial L}{\partial \dot{x}^\mu} = m g_{\rho\mu} \dot{x}^\rho \quad \mathbf{[0.5 pt]} \rightsquigarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) = m g_{\rho\mu} \ddot{x}^\rho + m g_{\rho\mu, \sigma} \dot{x}^\rho \dot{x}^\sigma \quad \mathbf{[0.5 pt]}. \quad (4)$$

For $\mu = r$, and using that the metric is diagonal, Eq. (2) thus gives

$$g_{rr} \ddot{r} = -g_{rr, \sigma} \dot{r} \dot{x}^\sigma + \frac{1}{2} \left(g_{tt, r} \dot{t}^2 + g_{rr, r} \dot{r}^2 + g_{\theta\theta, r} \dot{\theta}^2 + g_{\phi\phi, r} \dot{\phi}^2 \right) \quad (5)$$

$$= \frac{1}{2} \left(g_{tt, r} \dot{t}^2 - g_{rr, r} \dot{r}^2 + g_{\theta\theta, r} \dot{\theta}^2 + g_{\phi\phi, r} \dot{\phi}^2 \right) \quad (6)$$

$$F^{-1} \ddot{r} = \frac{1}{2} \left(-\frac{2GM}{r^2} \dot{t}^2 + F^{-2} \frac{2GM}{r^2} \dot{r}^2 + 2r \dot{\theta}^2 + 2r \sin^2 \theta \dot{\phi}^2 \right) \quad (7)$$

$$= -\frac{GM}{r^2} \dot{t}^2 + F^{-2} \frac{GM}{r^2} \dot{r}^2 + r \dot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2, \quad \mathbf{[1 pt]} \quad (8)$$

where we introduced the short-hand $F \equiv (1 - 2GM/r)$.

For $\mu = \theta$, we correspondingly obtain

$$g_{\theta\theta} \ddot{\theta} = -g_{\theta\theta, \sigma} \dot{\theta} \dot{x}^\sigma + \frac{1}{2} m g_{\rho\sigma, \theta} \dot{x}^\rho \dot{x}^\sigma = -g_{\theta\theta, r} \dot{\theta} \dot{r} + \frac{1}{2} m g_{\phi\phi, \theta} \dot{\phi}^2 \quad (9)$$

$$\ddot{\theta} = -\frac{2}{r} \dot{\theta} \dot{r} + \sin \theta \cos \theta \dot{\phi}^2. \quad \mathbf{[1 pt]} \quad (10)$$

b) A constant $\theta_0(\tau) = \pi/2$ clearly solves Eq. (10), with initial conditions $\theta(\tau_0) = \theta_0$ and $\dot{\theta}(\tau_0) = 0$ as stated in the question **[1 pt]**.

For circular orbits, we have $r = \text{const}$ and hence $\dot{r} = 0$ **[0.5 pt]**. With $\theta(\tau) = \pi/2$, Eq. (8) then becomes

$$0 = -\frac{GM}{r^2} \dot{t}^2 + r \dot{\phi}^2. \quad \mathbf{[0.5 pt]} \quad (11)$$

Hence, we find for the orbital frequency of circular orbits

$$\Omega = \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \sqrt{\frac{GM}{r^3}}. \text{ [1 pt]} \quad (12)$$

Notably, this agrees exactly with Kepler's third law, based on the Newtonian theory. **[1 bonus point.]**

c) For a circular orbit, we have $\dot{r} = \dot{\theta} = 0$ and hence $u^\mu = (\dot{t}, 0, 0, \dot{\phi}) = \dot{t}(1, 0, 0, \Omega)$ **[0.5 pt]**. Any (massive) object must move on a time-like trajectory, i.e. (choosing again $\theta = \pi/2$)

$$-1 = g_{\mu\nu}u^\mu u^\nu = g_{00}\dot{t}^2 + g_{\phi\phi}\dot{t}^2\Omega^2 = \dot{t}^2(-F + r^2\Omega^2). \quad (13)$$

In other words, we have

$$u^\mu \stackrel{\text{[1 pt]}}{=} \frac{1}{\sqrt{1 - 2GM/r - r^2\Omega^2}}(1, 0, 0, \Omega) \stackrel{\text{[0.5 pt]}}{=} \frac{1}{\sqrt{1 - 3GM/r}}(1, 0, 0, \Omega). \quad (14)$$

A stationary observer has a four-velocity $U^\mu = (\dot{t}, \mathbf{0}) = (-g_{00})^{-1/2}(1, \mathbf{0}) = (F^{-1/2}, \mathbf{0})$, such that contraction gives

$$u_\mu U^\mu = g_{00}u^0 U^0 = -\frac{\sqrt{F}}{\sqrt{F - r^2\Omega^2}}. \text{ [1 pt]} \quad (15)$$

In free-fall coordinates for someone at the position of the observer, on the other hand, we know that special relativity applies and we hence have

$$U^\mu = (1, \mathbf{0}), \quad u^\mu = \gamma(1, \mathbf{v}) \quad \rightsquigarrow \quad u_\mu U^\mu = -\gamma = -1/\sqrt{1 - v^2}. \text{ [1 pt]} \quad (16)$$

Equating these expressions, we can solve for

$$v = \sqrt{1 - (u \cdot U)^{-2}} = \sqrt{1 - \frac{F - r^2\Omega^2}{F}} = \frac{r\Omega}{\sqrt{1 - 2GM/r}} \text{ [0.5 pt]} \quad (17)$$

$$= \sqrt{\frac{GM/r}{1 - 2GM/r}}. \text{ [0.5 pt]} \quad (18)$$

d) As hinted in the exercise the frequency observed by a stationary observer A at the same distance r as the accretion disk is given by $\nu_A = \nu_0 \sqrt{1 - v^2}$. This relates to the frequency ν observed by a stationary far-away observer as

$$\frac{\nu}{\nu_A} = \frac{d\tau_A}{d\tau} = \sqrt{1 - 2GM/r}, \text{ [1 pt]} \quad (19)$$

where we used that the eigentime of a stationary observer is given by $d\tau = \sqrt{-ds^2} = \sqrt{-g_{00}}dt$. Putting things together, we hence obtain

$$\nu = \nu_0 \sqrt{1 - v^2} \sqrt{1 - 2GM/r} \text{ [0.5 pt]} \quad (20)$$

$$= \nu_0 \sqrt{1 - \frac{GM/r}{1 - 2GM/r}} \sqrt{1 - 2GM/r} = \nu_0 \sqrt{1 - 3GM/r}. \text{ [0.5 pt]} \quad (21)$$

Problem 3

The production of gravitational waves is proportional to the 2nd time derivative of the quadrupole tensor I_{ij} , so we first need to compute **[1 pt]**

$$I_{ij} \equiv \int d^3x x_i x_j T^{00}. \quad (22)$$

Let us choose a coordinate system with the collision point on the moon's surface as origo, such that the position of Oumuamua is given by $\mathbf{x}_O = (x_O, 0, 0)$. Approximating the object as point mass, its rest mass M contributes to the total energy density as **[1 pt]**

$$T^{00} = M\delta(x^2)\delta(x^3)\delta(x^1 - x_O) + T_{\text{moon}}^{00} \equiv T_O^{00} + T_{\text{moon}}^{00}. \quad (23)$$

Since $\int dx \delta(x)x^n = 0$ for $n = 1, 2$ (which is the integral we need to solve for $x = x^2$ and $x = x^3$ – but not for $x = x^1$), we thus get **[1 pt]**

$$I_{11}^O = Mx_O^2, \quad (24)$$

$$I_{ij}^O = 0 \quad \text{for all other } i, j, \quad (25)$$

and we are not interested in the quadrupole contribution of the moon, I_{ij}^{moon} , since $\dot{I}_{ij}^{\text{moon}} = 0$. In the trace-reversed gauge, the GW amplitude at the surface of the Earth is therefore given by **[1 pt]**

$$\bar{h}_{11} = \frac{2G}{d} \ddot{I}_{11} = \frac{4GM}{d} \partial_t(x_O \dot{x}_O) = \frac{4GM}{d} (\dot{x}_O^2 + x_O \ddot{x}_O), \quad (26)$$

where d is the distance to the moon.

For a constant deceleration a , both terms decrease monotonically, after the onset of the collision. In particular, starting from an initial velocity v_0 means that Oumuamua comes to full rest after a time $\Delta t = v_0/a$ and moves by $\Delta x = v_0 \Delta t - a(\Delta t)^2/2 = (v_0)^2/(2a)$ during the process. The initial (and hence maximal) value of the parenthesis is thus given by $(\dot{x}_I^2 + x_I \ddot{x}_I) = \frac{3}{2}v_0^2$, *independently of the acceleration rate* **[1 pt]**.

Plugging in numbers, with $M \approx 200 \times 25 \times 25 \text{m}^3 \times 5 \text{g/cm}^3 = 625 \cdot 10^6 \text{kg}$, and being careful with units (note in particular the factor of c^4 required to make the expression dimensionless), we thus expect an amplitude of **[1 pt for correct order of magnitude; 1 pt bonus for correct leading digit; -1 pt if result is not dimension-less (and this is not noticed)]**

$$\bar{h}_{11} = \frac{6 \times 6.7 \cdot 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2} \times 6.25 \cdot 10^8 \text{kg}}{4 \cdot 10^8 \text{m} \times (c = 3 \cdot 10^8 \text{m/s})^4} \times \left(\frac{5 \cdot 10^4 \text{km}}{\text{s}} \right)^2 \approx 2 \cdot 10^{-34}. \quad (27)$$

In order to be detectable – noting that the LIGO frequency range of 10-100 Hz is roughly in the correct ballpark – we'd thus need a strain, and therefore mass, that is larger by about a factor of 10^{11} , $M \sim 10^{19} \text{kg}$ **[1 pt]**, and hence still slightly lighter than the heaviest known asteroids in the solar system ($\sim 10^{20} \text{kg}$).

Problem 4

a) From the FRW metric, the proper distance along a circle (with $dt = dr = d\phi = 0$)

is $ds = ar d\theta$ **[1 pt]**. We therefore have $D = a_e r_e \delta\theta$, where we have to take the scale factor at emission since it is the proper size that matters. In other words, $d_A = r_e(1+z)^{-1}$ **[1 pt]**.

b) The telescope is located at the surface of a sphere around the emitting object, with total area (today) of $A = 4\pi(a_0 r_e)^2$; this means that the luminosity distance is given by $d_L = r_e \sqrt{L/L_{\text{int}}^0}$, where L is the total emitted power at time t_e and L_{int}^0 is the total, integrated power at a (coordinate) distance of r_e from the source today **[1 pt]**. The ratio L/L_{int}^0 decreases with one factor of $(1+z)$ each because *i)* the photons are redshifted and *ii)* the number of photons decreases due to the expansion (the number density decreases as a^{-3} , hence the the number of photons crossing A decreases as a^{-1}) **[0.5 pt for each argument]**. In total, we thus have $d_L = r_e(1+z)$ and hence $n = 2$ **[1 pt]**.

c) For photons propagating in a flat FRW spacetime, we have $0 = ds^2 = -dt^2 + a^2 d\mathbf{x}^2$ **[0.5 pt]** and hence

$$r_e = \int_0^{r_e} dr = \int_{t_e}^{t_0} \frac{dt}{a(t)} \quad \text{[0.5 pt]} \quad (28)$$

$$= \int_{a_e}^1 \frac{da}{a\dot{a}} = \int_0^{z_e} \frac{dz}{(1+z)\dot{a}(z)} \cdot \text{[1 pt]} \quad (29)$$

With

$$d_L = r_e(1+z_e) = (1+z_e) \int_0^{z_e} \frac{dz}{H(z)}, \quad (30)$$

and denoting derivatives with respect to z_e with $'$, we then get

$$d'_L = \frac{1}{\dot{a}(z_e)} + \int_0^{z_e} \frac{dz}{(1+z)\dot{a}(z)} \quad \text{[0.5 pt]} \quad (31)$$

$$d''_L = -\frac{\ddot{a}(z_e)}{\dot{a}^2(z_e)} \underbrace{\frac{dt}{dz_e}}_{(d(a^{-1}-1)/dt)^{-1}} + \frac{1}{(1+z_e)\dot{a}(z_e)} \quad (32)$$

$$= \frac{\ddot{a}(z_e)a^2(z_e)}{\dot{a}^3(z_e)} + \frac{1}{(1+z_e)\dot{a}(z_e)} \cdot \text{[1 pt]} \quad (33)$$

Taylor-expanding around $z_e = 0$ therefore gives

$$d_L = z_e \frac{1}{\dot{a}_0} + \frac{1}{2} z_e^2 \left(\frac{\ddot{a}_0}{\dot{a}_0^3} + \frac{1}{\dot{a}_0} \right). \quad (34)$$

From this we can read off $A = 1/\dot{a}_0$ **[0.5 pt]** and $B = 1 + \ddot{a}_0/\dot{a}_0^2$ **[0.5 pt]**. Since $a_0 = 1$, we thus have $A = H_0^{-1}$ – which is *Hubble's law* **[0.5 pt]**.

(B , instead, is typically parameterized as $B = 1 - q_0$, where the *deceleration parameter* $q_0 \equiv -(\ddot{a}a/\dot{a}^2)|_{t_0}$ contains valuable information about the energy content of the universe – and is in practice the main way to determine the value of the cosmology constant, i.e. the vacuum energy density, from supernova observations.)