

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: FYS4160 General relativity
Day of exam: 10. June 2026
Exam hours: 9.00-13.00, 4 hours
This examination paper consists of 3 pages
Permitted materials: Three sheets (6 pages) A4 paper of notes

Language: The solutions may be written in Norwegian or English depending on your own preference.

*Make sure that your copy of this examination paper is complete before answering.
All answers should be justified*

PROBLEM 1

Unusual geodesics in the Schwarzschild geometry

The Schwarzschild geometry is given by the line element

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{1 - 2GM/r} dr^2 + r^2 d\Omega^2$$

and describes the spacetime around a point mass M .

- a) How many Killing vectors are there in the Schwarzschild geometry? Describe the symmetries corresponding to these Killing vectors, and give explicit expressions for two of the Killing vectors in the coordinate basis.
- b) Explain why the quantities

$$E = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau} \quad \text{and} \quad L = r^2 \frac{d\phi}{d\tau}$$

where τ is proper time, are conserved along geodesic curves.

- c) Show that the radial motion of a free falling particle is equivalent to the non-relativistic motion of a particle of unit mass in the potential

$$U(r) = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}.$$

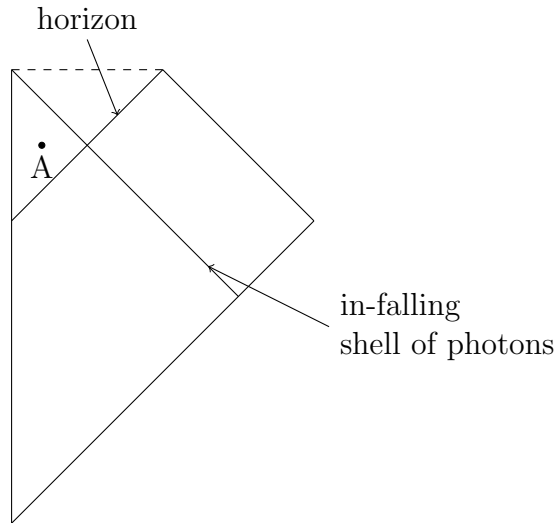
- d) Explain how one could have a particle falling in from infinity, circle the black hole once (not at a constant radius) and then continue to infinity with the final asymptotic velocity parallel to the initial velocity. Explain what property of the potential $U(r)$

makes this possible in the relativistic case while there are no similar trajectories in Newtonian gravity.

- e) The once-encircling trajectory described in the previous question can only exist for angular momenta satisfying $L > \beta GM$ for some value of the dimensionless number β . Demonstrate this and find the value of β .
- f) Can one have trajectories circling the black hole more than once before continuing to infinity? You are not required to find any expression for the trajectories, but give a clear explanation supported by estimates justifying existence or non-existence of trajectories with a given number of circlings.

PROBLEM 2

Conformal diagram of a collapsing shell



The figure above shows the conformal diagram of a thin spherical, non-rotating shell of radiation collapsing to form a black hole.

- a) Indicate in the diagram where you will find past and future timelike infinity, past and future null infinity and spacelike infinity.
- b) For a free-falling observer at the event A, describe the gravitational field that this observer will observe at that time.
- c) Describe possible future worldlines of massive particles starting from the event A. Which future infinities can be reached? Illustrate your answer in the diagram.

PROBLEM 3

Energy loss from the Earth's orbit due to gravitational waves

An accelerated mass, like a planet in orbit around a star, will emit gravitational waves and thereby lose energy. We will calculate this energy loss and estimate the effect of this on the Earth's orbit.

- a) Show that the quadrupole tensor of a planet of mass m in a circular orbit of radius R around a star of mass M takes the form

$$\begin{aligned}I_{xx} &= \alpha \cos^2 \omega t \\I_{yy} &= \alpha \sin^2 \omega t \\I_{xy} &= \alpha \cos \omega t \sin \omega t\end{aligned}$$

where ω is the orbital angular frequency, and determine the parameter α . Assume that $M \gg m$.

- b) Describe the polarization of the gravitational wave emitted in the direction perpendicular to the orbital plane of the planet.
- c) Show the radiated power of gravitational waves is given by an expression of the form $P = \beta m^2 R^4 \omega^6$ and determine the parameter β .
- d) Estimate the time until the Earth will hit the Sun and evaluate an approximate numerical estimate from your expressions (The mass of the Sun is $M = 2 \cdot 10^{30}$ kg, the mass of the Earth is $m = 6 \cdot 10^{24}$ kg and the distance from the Sun to the Earth is $R_0 = 1.5 \cdot 10^{11}$ m. The gravitational constant is $G = 6.67 \cdot 10^{-11}$ m³/kg s².)

PROBLEM 4

Comoving coordinates

In cosmology, we usually use comoving coordinates where the coordinates of mass points of the cosmic fluid are time independent. Such coordinates can also be constructed in more general cases, where the spatial geometry is not homogeneous or isotropic.

Assume the metric has the form ($i, j = 1, 2, 3$ are spatial indices)

$$ds^2 = -dt^2 + g_{ij} dx^i dx^j.$$

Show that comoving (stationary) observers are free-falling and that t is the proper time of such observers.