

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: FYS4160 General relativity
Day of exam: 10. June 2026
Exam hours: 9.00-13.00, 4 hours
This examination paper consists of 10 pages
Permitted materials: Three sheets (6 pages) A4 paper of notes

Language: The solutions may be written in Norwegian or English depending on your own preference.

*Make sure that your copy of this examination paper is complete before answering.
All answers should be justified*

PROBLEM 1

Unusual geodesics in the Schwarzschild geometry

The Schwarzschild geometry is given by the line element

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{1 - 2GM/r} dr^2 + r^2 d\Omega^2$$

and describes the spacetime around a point mass M .

- a) How many Killing vectors are there in the Schwarzschild geometry? Describe the symmetries corresponding to these Killing vectors, and give explicit expressions for two of the Killing vectors in the coordinate basis.

Solution: There are 4 Killing vectors, corresponding to time translation and rotations around the three coordinate axes. The metric components in the given coordinates are independent of t and ϕ and the corresponding tangent vectors are Killing vectors. They are for time translation $K^\mu = (\partial_t)^\mu = (1, 0, 0, 0)$ and for rotation around the z -axis $R^\mu = (\partial_\phi)^\mu = (0, 0, 0, 1)$.

- b) Explain why the quantities

$$E = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau} \quad \text{and} \quad L = r^2 \frac{d\phi}{d\tau}$$

where τ is proper time, are conserved along geodesic curves.

Solution: The conserved quantities corresponding to the Killing vectors are given in terms of the four-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ as

$$E = -K_\mu u^\mu = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau}$$

and

$$L = R_\mu u^\mu = r^2 \frac{d\phi}{d\tau}.$$

- c) Show that the radial motion of a free falling particle is equivalent to the non-relativistic motion of a particle of unit mass in the potential

$$U(r) = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}.$$

Solution: We know that the trajectory will lie in a plane, and choose this to be the $\theta = \pi/2$ -plane. We then have that

$$g_{\mu\nu} u^\mu u^\nu = -\left(1 - \frac{2GM}{r}\right) \dot{t}^2 + \frac{1}{1 - 2GM/r} \dot{r}^2 + r^2 \dot{\phi}^2 = -1$$

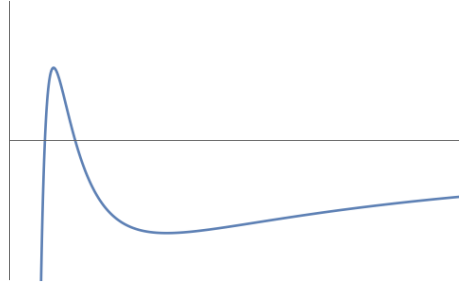
where $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. Using the conserved quantities to eliminate $\dot{t}$ and $\dot{\phi}$ we get

$$\frac{1}{2} \dot{r}^2 + U(r) = \frac{1}{2} E^2 - \frac{1}{2} = \mathcal{E}$$

which is the conservation of energy of the non-relativistic motion of a particle of unit mass in the potential $U(r)$.

- d) Explain how one could have a particle falling in from infinity, circle the black hole once (not at a constant radius) and then continue to infinity with the final asymptotic velocity parallel to the initial velocity. Explain what property of the potential $U(r)$ makes this possible in the relativistic case while there are no similar trajectories in Newtonian gravity.

Solution: The potential $U(r)$ has the form



A particle with an energy that is a little bit less than the peak in the potential will turn very slowly, and therefore spend a long time close to the minimal radius. This gives it time to circle the black hole before returning to infinite r . The exact amount of circling, and therefore the exit direction, will depend on the energy. The closer to the peak, the longer it will circle. Choosing the right energy, we can get it to exit in the same asymptotic direction as it came in. In Newtonian gravity, the potential is $-\frac{GM}{r} + \frac{L^2}{2r^2}$, so the term $-\frac{GML^2}{r^3}$ which causes the potential to go to $-\infty$ at small r is absent. So the potential does not have any maximum, but rises to $+\infty$ at small r . Therefore, there are no trajectories with very slow turning of the radial motion at the minimal r , and therefore no such circling trajectories.

- e) The once-encircling trajectory described in the previous question can only exist for angular momenta satisfying $L > \beta GM$ for some value of the dimensionless number β . Demonstrate this and find the value of β .

Solution: Since $\lim_{r \rightarrow \infty} U(r) = 0$ a particle coming in from infinity must have a positive energy. Therefore the turning and return to infinity can only happen if the peak of the potential is at positive energy. When L is too small, the peak is at negative energy (or disappears completely). To find the minimal L , we require that $U(r) = 0$ and $U'(r) = 0$. This gives two equations for the unknowns r and L which we solve to find $L = 4GM$.

- f) Can one have trajectories circling the black hole more than once before continuing to infinity? You are not required to find any expression for the trajectories, but give a clear explanation supported by estimates justifying existence or non-existence of trajectories with a given number of circlings.

Solution: As discussed above, the closer the energy is to the peak of the potential, the longer time it will spend in turning the radial motion, and the more time it will have for circling. We will show that there is no limit to the number of circlings that can take place. Let r_- be the position of the maximum of the potential and r_0 be the closest approach of the trajectory. Let $\bar{r} > r_0$ be any fixed radius outside the turning point. The equation of motion is

$$\frac{1}{2}\dot{r}^2 + U(r) = \mathcal{E},$$

which we integrate from the turning point r_0 to get the time $\bar{t}$ to reach the fixed radius $\bar{r}$,

$$\bar{t} = \int_{r_0}^{\bar{r}} \frac{dr}{\sqrt{2(\mathcal{E} - U(r))}}.$$

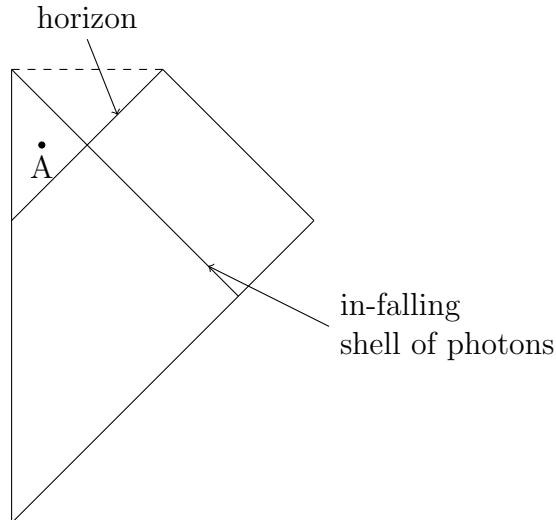
Close to r_- we approximate the potential to second order, $U(r) \approx U(r_-) - A(r - r_-)^2$ for some constant A . Then $\mathcal{E} - U(r) \approx A[(r - r_-)^2 - (r_0 - r_-)^2]$, and changing integration variable to $x = r - r_-$ we get

$$\bar{t} = \frac{1}{\sqrt{2A}} \int_{x_0}^{\bar{x}} \frac{dx}{\sqrt{x^2 - x_0^2}} = \frac{1}{\sqrt{2A}} \int_1^{\bar{u}} \frac{du}{\sqrt{u^2 - 1}}$$

where $\bar{x} = \bar{r} - r_-$, $x_0 = r_0 - r_-$, $u = x/x_0$ and $\bar{u} = \bar{x}/x_0$. As $r_0 \rightarrow r_-$ we have $x_0 \rightarrow 0$ and $\bar{u} \rightarrow \infty$. At large u the integrand is approximately $1/u$, and we get that $\bar{t} \approx \ln \bar{u}$ which diverges as $\bar{u} \rightarrow \infty$. Since $\dot{\phi} = L/r^2$ is approximately constant during this time (as long as we choose $\bar{r}$ close to r_0) the total angle rotated, $\Delta\phi = \int_0^{\bar{t}} \dot{\phi} dt$, can be arbitrary large.

PROBLEM 2

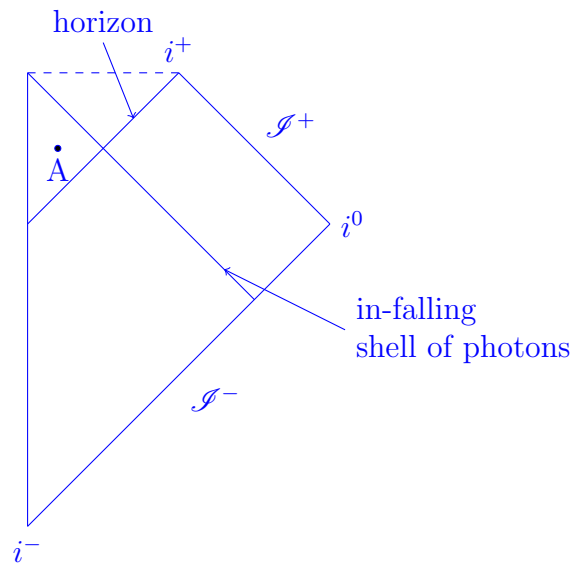
Conformal diagram of a collapsing shell



The figure above shows the conformal diagram of a thin spherical, non-rotating shell of radiation collapsing to form a black hole.

- Indicate in the diagram where you will find past and future timelike infinity, past and future null infinity and spacelike infinity.

Solution:



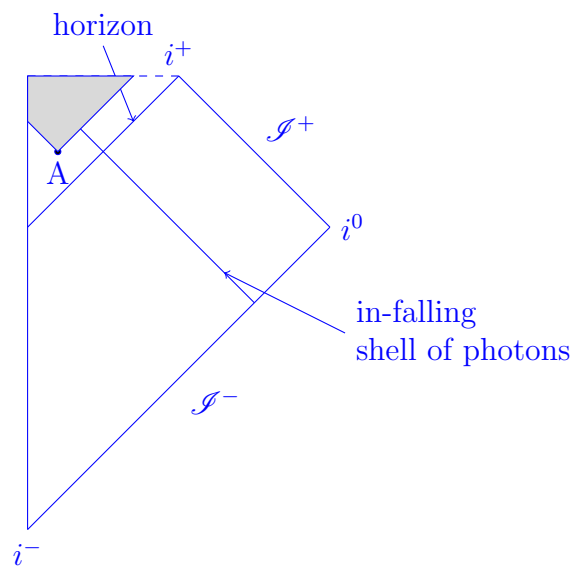
- b) For a free-falling observer at the event A, describe the gravitational field that this observer will observe at that time.

Solution: Being inside the collapsing shell, there will be no gravitational effect. Spacetime is Minkowski.

- c) Describe possible future worldlines of massive particles starting from the event A. Which future infinities can be reached? Illustrate your answer in the diagram.

Solution:

The shaded region shows the future lightcone of the point A. All possible future worldlines lie within the lightcone. No asymptotic future infinities can be reached, the worldlines will end at the singularity.



PROBLEM 3

Energy loss from the Earth's orbit due to gravitational waves

An accelerated mass, like a planet in orbit around a star, will emit gravitational waves and thereby lose energy. We will calculate this energy loss and estimate the effect of this on the Earth's orbit.

- a) Show that the quadrupole tensor of a planet of mass m in a circular orbit of radius R around a star of mass M takes the form

$$\begin{aligned} I_{xx} &= \alpha \cos^2 \omega t \\ I_{yy} &= \alpha \sin^2 \omega t \\ I_{xy} &= \alpha \cos \omega t \sin \omega t \end{aligned}$$

where ω is the orbital angular frequency, and determine the parameter α . Assume that $M \gg m$.

Solution: For all planets around normal stars, the orbit will be very close to the Newtonian orbit, so we can use this when finding the motion of the planet. Let the z -axis be normal to the orbital plane. The position of the planet is

$$\vec{r}_p(t) = R(\cos \omega t, \sin \omega t, 0)$$

where $\omega = \sqrt{\frac{GM}{R^3}}$ is the orbital angular frequency. The energy density is $T_{00}(\vec{r}) = m\delta(\vec{r} - \vec{r}_p)$ from which we find the nonzero components of the quadrupole tensor

$$\begin{aligned} I_{xx} &= m \int d^3r x^2 \delta(\vec{r} - \vec{r}_p) = mR^2 \cos^2 \omega t = \frac{1}{2}mR^2(1 + \cos 2\omega t), \\ I_{yy} &= mR^2 \sin^2 \omega t = \frac{1}{2}mR^2(1 - \cos 2\omega t), \\ I_{xy} &= mR^2 \cos \omega t \sin \omega t = \frac{1}{2}mR^2 \sin 2\omega t. \end{aligned}$$

- b) Describe the polarization of the gravitational wave emitted in the direction perpendicular to the orbital plane of the planet.

Solution: The trace reversed metric perturbation is given by

$$\bar{h}_{ij} = \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}(t_r) = -\frac{4GmR^2\omega^2}{r} \begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To find the polarization, we should convert to the transverse traceless gauge, but in this case it is already on that form. The wave is an equal combination of $+$ - and $\times$ -polarized waves. Since there is a 90° phase shift, it means that it is circularly polarized.

- c) Show the radiated power of gravitational waves is given by an expression of the form $P = \beta m^2 R^4 \omega^6$ and determine the parameter β .

Solution: The general expression for radiated power is

$$P = \frac{G}{5} \langle \ddot{J}_{ij} \ddot{J}^{ij} \rangle$$

with the reduced quadrupole tensor

$$J_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} I_k^k = \frac{1}{2} m R^2 \omega^3 \begin{pmatrix} \frac{1}{3} + \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & \frac{1}{3} - \cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We then get

$$P = \frac{16G}{5} m^2 R^4 \omega^6 \langle 2 \sin^2 2\omega t + 2 \cos^2 2\omega t \rangle.$$

Since $\langle \sin^2 2\omega t \rangle = \langle \cos^2 2\omega t \rangle = \frac{1}{2}$ we get

$$P = \frac{32G}{5} m^2 R^4 \omega^6.$$

- d) Estimate the time until the Earth will hit the Sun and evaluate an approximate numerical estimate from your expressions (The mass of the Sun is $M = 2 \cdot 10^{30}$ kg, the mass of the Earth is $m = 6 \cdot 10^{24}$ kg and the distance from the Sun to the Earth is $R_0 = 1.5 \cdot 10^{11}$ m. The gravitational constant is $G = 6.67 \cdot 10^{-11}$ m³/kg s².)

Solution: Approximating the motion of the planet as Newtonian, we have that the energy of a circular orbit of radius R is

$$E_c = -\frac{GMm}{2R}.$$

Equating the rate of energy change with the radiated power, we get

$$\dot{E}_c = \frac{GMm}{2R^2} \dot{R} = -P = -\frac{32G}{5} m^2 R^4 \omega^6 = -\frac{32G^4}{5} m^2 M^3 \frac{1}{R^5},$$

where we used that $\omega = \sqrt{\frac{GM}{R^3}}$. We rewrite this as

$$\dot{R} = -\frac{64G^3}{5} m M^2 \frac{1}{R^3}.$$

With the initial radius as R_0 and the radius when the planet merges with the star as R_m , we solve this to get the time of merger as

$$t_m = \frac{5}{256G^3 m M^2} (R_0^4 - R_m^4).$$

For the Earth at present $R_0 \gg R_m$ and we have approximately

$$t_m \approx \frac{5c^5 R_0^4}{256G^3 m M^2} \approx 10^{23} \text{ years.}$$

where we retored the appropriate factors of c by dimensional analysis.

PROBLEM 4

Comoving coordinates

In cosmology, we usually use comoving coordinates where the coordinates of mass points of the cosmic fluid are time independent. Such coordinates can also be constructed in more general cases, where the spatial geometry is not homogeneous or isotropic.

Assume the metric has the form ($i, j = 1, 2, 3$ are spatial indices)

$$ds^2 = -dt^2 + g_{ij} dx^i dx^j.$$

Show that comoving (stationary) observers are free-falling and that t is the proper time of such observers.

Solution: A stationary observer has position $x^\mu = (t, x^i(0))$ which gives the four-velocity $u^\mu = (\frac{dt}{d\tau}, 0, 0, 0)$ where τ is proper time. This should be normalized so that $-1 = g_{\mu\nu} u^\mu u^\nu = g_{00} \left(\frac{dt}{d\tau}\right)^2$ which gives $\frac{dt}{d\tau} = 1$, so t is proper time. The geodetic equation is

$$\frac{du^\mu}{d\tau} + \Gamma_{\alpha\beta}^\mu u^\alpha u^\beta = 0$$

where the Christoffel symbols are

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2}g^{\mu\lambda}(\partial_{\alpha}g_{\beta\lambda} + \partial_{\beta}g_{\alpha\lambda} - \partial_{\lambda}g_{\alpha\beta})$$

For stationary observers, the relevant components are

$$\Gamma_{00}^{\mu} = \frac{1}{2}g^{\mu\lambda}(2\partial_0g_{0\lambda} - \partial_{\lambda}g_{00}) = 0.$$

Since $\frac{du^{\mu}}{d\tau} = 0$ for comoving observers, they satisfy the geodesic equation and are therefore free-falling.