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http://www.uio.no/studier/emner/matnat/fys/FYS4170/h14/

Mid-term exam

Lecture autumn 2014: Relativistic quantum field (FYS4170)

⇝ *Answers must be handed in latest* **Tuesday, 28 October 2014, 10:15am**; *you can use the corresponding box (labelled with the course name and code) at the administrative office of the Physics Department. Please write your candidate number (not name!) on the top of the front page.*
The problems are – very roughly – ordered by increasing difficulty. Maximal number of available points: 50.

Problem 1

A useful basis representation for any 4×4 matrix in spinor space is given by $\Gamma_j \in \{\mathbf{1}, i\gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$, where $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. As we have shown in the lecture, in particular, any *Dirac bilinear* of the form $\overline{\psi}_f \Gamma_j \psi_i$ has then a definite transformation property under the Lorentz group. In particular, this allows to define transition currents $j^\mu_{fi} \equiv \overline{\psi}_f \gamma^\mu \psi_i$ (vector) and $j^{\mu 5}_{fi} \equiv \overline{\psi}_f \gamma^\mu \gamma^5 \psi_i$ (axial vector).

a) Show that all these bilinears are real for $i = f$, i.e. $\left(\overline{\psi}\Gamma_j\psi\right)^* = \overline{\psi}\Gamma_j\psi$ for all Γ_j. Why is that an advantage? (*6 points*)

b) Prove the *Gordon identity*, which states that the vector coupling can be decomposed into a “current coupling” and a “dipole moment coupling” in the following sense: (*4 points*)

$$(m_f + m_i) j^\mu_{fi} = (p_f + p_i)^\mu \overline{\psi}_f \psi_i + i(p_f - p_i)_\nu \overline{\psi}_f \sigma^{\mu\nu} \psi_i$$

[Hint: To show this, rewrite the tensor bilinear by using the Dirac equation. p_i and p_f are the 4-momenta of the initial and final states.]

Problem 2

The field operator ψ for the quantized Dirac field can be written as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_s \left(a^s_\mathbf{p} u^s(p) e^{-ip\cdot x} + b^{s\,\dagger}_\mathbf{p} v^s(p) e^{ip\cdot x} \right) .$$

a) What are the meaning and the main properties of the various components of this expression? (*5 points*)

b) Recap, in your own words, how we arrived at this expression, and state the most important intermediate results! [don't use more than half a page on this!] (*5 points*)

Problem 3

This problem considers the QED process of electron-positron scattering, $e^+e^- \to e^+e^-$, also known as *Bhabha scattering*.

a) Draw the two Feynman diagrams that describe this process to lowest order – i.e. $\mathcal{O}(\alpha_{\mathrm{em}}^2)$ in the cross section – and write down the corresponding amplitudes $\mathcal{M}_i$ for each diagram ($i = 1, 2$) in momentum representation (*5 points*)

b) When 'adding' these two diagrams to the total amplitude, $\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$, why must there be a minus sign? (*2 points*)

c) Going to the next order in perturbation theory (i.e. which order in α_{em}?), how many Feynman diagrams contribute to the invariant matrix element $\mathcal{M}$? Draw all these diagrams and indicate the relative sign with which they contribute. (*8 points*) *[Hint: Recall that only fully contracted, amputated diagrams contribute to $\mathcal{M}$!]*

Problem 4

In order to describe the scattering of an electron, or positron, in a time-independent classical electromagnetic field one can simply replace the QED vertex rule $-ie\gamma^\mu \to -ie\gamma^\mu \tilde{A}_\mu(\mathbf{q})$. Here, $A_\mu(\mathbf{x})$ is the classical electromagnetic potential, $\tilde{A}_\mu(\mathbf{q})$ its Fourier transform and $q \equiv p_f - p_i$ the difference between incoming and outgoing *fermion* momenta. For a potential that is not only time-independent but also localized in space, the scattering cross section can then be written as

$$d\sigma = \frac{1}{2|\mathbf{p}_i|} \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \left|\mathcal{M}(p_i \to p_f)\right|^2 (2\pi)\delta(E_f - E_i)\,.$$

[See P&S, problem 4.4, for a motivation of all these statements. Note that the amplitude is not dimensionless in this case, like for $2 \to 2$ scattering!]

a) Show that the last expression is equivalent to (*2 points*)

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} \left|\mathcal{M}(p_i \to p_f)\right|^2$$

b) Compute the scattering amplitude $\mathcal{M}$ for the scattering of an electron in the Coulomb potential created by a nucleus of charge Z, i.e. $A^\mu = (Ze/4\pi r, \mathbf{0})$. How does this expression look like for the scattering of a positron? (*5 points*)

c) Using the above expressions, calculate the spin-averaged cross-section for the scattering of an electron in a Coulomb potential. The result is known as the *Mott formula*. Take the non-relativistic limit of this expression to obtain an expression earlier obtained by *Rutherford*. (*8 points*)

Problem X

(This problem is not formally part of the exam; answering it, however, will give bonuspoints that are added to the score achieved in solving problems 1–4.)

The stress-energy tensor is the conserved current corresponding to the invariance of any Lagrangian $\mathcal{L}$ under space-time translations. Likewise, angular momentum conservation follows from the invariance under rotations.

In this discussion, the only Poincaré transformations that are missing are Lorentz boosts. What is the conserved quantity that corresponds to the invariance of any physical theory to this particular type of Lorentz transformations? Use Noether's theorem to derive the answer. *(Up to 10* bonus*points)*