University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 10

Problem 1 Contractions for fermion fields (J. Skaar)

a) Wick's theorem for two fermion fields is

$$T\psi(x)\bar{\psi}(y) = N\psi(x)\bar{\psi}(y) + \overbracket{\psi(x)\bar{\psi}}(y). \tag{1}$$

If (1) is taken as the definition of the contraction $\overbracket{\psi(x)\bar{\psi}}(y)$, and

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle, \tag{2}$$

prove that

$$\begin{aligned}\overbracket{\psi(x)\bar{\psi}}(y) &= \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = S_F(x-y)\\ &= \begin{cases}\{\psi^+(x), \bar{\psi}^-(y)\}, & x^0 > y^0,\\ -\{\bar{\psi}^+(y), \psi^-(x)\}, & x^0 < y^0,\end{cases}\\ &= -\overbracket{\bar{\psi}(y)\psi}(x),\end{aligned} \tag{3}$$

and

$$\overbracket{\psi(x)\psi}(y) = 0 = \overbracket{\bar{\psi}(x)\bar{\psi}}(y) \tag{4}$$

(Strictly speaking, we need spinor indices on the spinors above to make sense of the equations).

Solution: The first equality follows by taking the vacuum expectation value of (1), since the vacuum expectation value of normal ordered operators is zero. For the last equality, note that for $x^0 > y^0$,

$$\begin{aligned}T\psi(x)\bar{\psi}(y) = \psi(x)\bar{\psi}(y) &= \psi^+(x)\bar{\psi}^+(y) + \psi^+(x)\bar{\psi}^-(y)\\ &\quad + \psi^-(x)\bar{\psi}^+(y) + \psi^-(x)\bar{\psi}^-(y).\end{aligned} \tag{5}$$

Taking the vacuum expectation value (and taking into account the usual expressions for $\psi(x)$ and $\bar{\psi}(y)$ in terms of ladder operators), we obtain

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \langle 0|\psi^+(x)\bar{\psi}^-(y)|0\rangle = \{\psi^+(x), \bar{\psi}^-(y)\}. \tag{6}$$

When $x^0 < y^0$ we have $T\psi(x)\bar{\psi}(y) = -\bar{\psi}(y)\psi(x)$, and proceed in the same way.

b) For all examples in P&S (see e.g. the section about Yukawa theory in Ch. 4), none of them has contractions between two external states (between two initial-state particles, between two final-state particles, or between initial-state and final-state particles). Can you explain why this possibility is ignored?

Problem 2 Interaction between fermions and a classical electromagnetic field

a) Peskin & Schroeder problem 4.4a p. 129. Note that your result can be represented as a Feynman diagram with associated rule (see 4.4b).

Solution: Write $S = 1 + iT$ (and always keep the i in iT to make it different from the time ordering symbol T):

$$\begin{aligned}\langle p'|iT|p\rangle &= {}_0\langle p'|Te^{-i\int dtH_I(t)}|p\rangle_{0,\text{connected, amputated}} \\ &= {}_0\langle p'|Te^{-i\int d^4x e\bar{\psi}\gamma^\mu\psi A_\mu}|p\rangle_{0,\text{connected, amputated}}.\end{aligned} \tag{7}$$

Here we have used

$$H_I(t) = \int d^3x e\bar{\psi}\gamma^\mu\psi A_\mu, \tag{8}$$

where A_μ represent the classical, electromagnetic field, and ψ is the interaction-picture (quantized) Dirac field.

To first order:

$$\begin{aligned}\langle p'|iT|p\rangle &= -ie\int d^4x A_\mu \cdot {}_0\langle p'|T\bar{\psi}\gamma^\mu\psi|p\rangle_0 = -ie\int d^4x A_\mu \cdot {}_0\langle p'|\bar{\psi}\gamma^\mu\psi|p\rangle_0 \\ &= -ie\int d^4x A_\mu(x)e^{i(p'-p)x}\bar{u}(p')\gamma^\mu u(p) \\ &= -ie\tilde{A}_\mu(p'-p)\bar{u}(p')\gamma^\mu u(p).\end{aligned} \tag{9}$$

To obtain the second line, we use that the contraction between $\psi(x)$ and $|p\rangle_0$ gives $u(p)e^{-ipx}$. The other contraction gives $\bar{u}(p')e^{ip'x}$. Note the presence of the exponentials. (In the momentum-space Feynman rules for QED or Yukawa theory, the exponentials are omitted, but that is because they haved been used together with $\int d^4x$ to obtain momentum-conserving delta functions.)

We note that (9) can be represented by the Feynman diagram and Feynman rule mentioned in 4.4b.

b) In what sense is 4-momentum conserved in this process, in the general case and in the special case where the classical field is independent of time?

Solution: We have $p' = p + q$, i.e., the output momentum is equal to the input momentum plus the momentum from the Fourier component of the classical field. In the general case the classical field consists of a spectrum of Fourier components with momenta q; this leads to a spectrum of output momenta p' even though p is fixed. When the classical field is independent of time the Fourier transform contains a delta function $\delta(q^0)$ so the input and output energies of the fermion are equal.

Problem 3 Spinor algebra and trace methods (T. Klungland)

a) For any operator Γ consisting of a product of an arbitrary number of γ matrices, and any two spinors $v^s(p)$, $u^r(k)$ (r and s label the spin states of the two spinors) (whether the spinors are particle or antiparticle spinors is irrelevant; these are just chosen as an example), show that

$$(\overline{v}^s(p)\Gamma u^r(k))^\dagger = \overline{u}^r(k)\Gamma' v^s(p), \tag{10}$$

where $\overline{u} = u^\dagger\gamma^0$ and Γ' is the same product as Γ with the order of matrices reversed. You will need the identity $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

Solution: Writing Γ as $\Gamma = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}$, its hermitian conjugate is given by

$$\begin{aligned}
\Gamma^\dagger &= (\gamma^{\mu_n})^\dagger\cdots(\gamma^{\mu_2})^\dagger(\gamma^{\mu_1})^\dagger \\
&= \gamma^0\gamma^{\mu_n}\gamma^0\cdots\gamma^0\gamma^{\mu_2}\gamma^0\gamma^0\gamma^{\mu_1}\gamma^0 \\
&= \gamma^0\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\gamma^0 \\
&= \gamma^0\Gamma'\gamma^0,
\end{aligned}$$

where the third equality follows since all "interior" γ^0 matrices are in pairs, all of which multiply to 1. Using this, our desired result follows almost immediately:

$$\begin{aligned}
(\overline{v}^s(p)\Gamma u^r(k))^\dagger &= \left(v^{s\dagger}(p)\gamma^0\Gamma u^r(k)\right)^\dagger \\
&= u^{r\dagger}(k)\Gamma^\dagger\gamma^0 v^s(p) \\
&= \overline{u}^r\Gamma' v^s(p).
\end{aligned}$$

b) Suppose that the scattering of a fermion off of some potential is described by the matrix element

$$i\mathcal{M} = iB_\mu\overline{u}^r(p')\gamma^\mu u^s(p),$$

where p and p' describe the initial and final momenta, respectively, and s and r the initial and final spins. B_μ contains the other numerical factors, that are unimportant for our purposes. Use the spin sum relation in Eq. (3.66) in P&S to show that the unpolarized, spin-averaged squared matrix element (this means that we average over the 2 possible initial-state spins and sum over the final-state ones) is given by

$$\left\langle|\mathcal{M}|^2\right\rangle = \frac{1}{2}\sum_{s=1}^{2}\sum_{r=1}^{2}\left[B_\mu\overline{u}^r(p')\gamma^\mu u^s(p)\right]^\dagger\left[B_\nu\overline{u}^r(p')\gamma^\nu u^s(p)\right] \tag{11}$$

$$= \frac{1}{2}B_\mu^\dagger B_\nu \mathrm{Tr}\left[(\not{p}' + m)\gamma^\mu(\not{p} + m)\gamma^\nu\right], \tag{12}$$

where m is the mass of the fermion and $\not{p} = p_\mu\gamma^\mu$.

Hint: One way of doing this is explained on p. 132 in P&S, but it is easier to use a) that a product of matrices that gives a scalar, or a 1×1 matrix can be written as its own trace, and b) the cyclic and linear properties of traces.

Solution: Using the result of part a, we first have

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{2} B_\mu^\dagger B_\nu \sum_{s=1}^{2} \sum_{r=1}^{2} \overline{u}^s(p) \gamma^\mu u^r(p') \overline{u}^r(p') \gamma^\nu u^s(p).$$

Next we note that $\overline{u}^s(p)\gamma^\mu u^r(p')\overline{u}^r(p')\gamma^\nu u^s(p) = \mathrm{Tr}[\text{same expression}]$, and use the cyclic property of traces to move $u^s(p)$ from the back to the front; moving the sums into the trace (which we can do because of the linearity of traces) and using the aforementioned spin sum relation, we find

$$\begin{aligned}\left\langle |\mathcal{M}|^2 \right\rangle &= \frac{1}{2} B_\mu^\dagger B_\nu \mathrm{Tr}\left[\left(\sum_{s=1}^{2} u^s(p) \overline{u}^s(p) \right) \gamma^\mu \left(\sum_{r=1}^{2} u^r(p') \overline{u}^r(p') \right) \gamma^\nu \right] \\ &= \frac{1}{2} B_\mu^\dagger B_\nu \mathrm{Tr}\left[(\not{p}' + m) \gamma^\mu (\not{p} + m) \gamma^\nu \right].\end{aligned}$$