

University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 2

Problem 1 Two scalar fields (T. Bringmann)

Derive the equations of motion for two interacting, real, scalar fields ϕ_1 and ϕ_2 , described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) - \frac{1}{2}m_1^2 \phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2}m_2^2 \phi_2^2 - \lambda \phi_1^2 \phi_2^2,$$

where λ is a constant. Do so by directly using the principle of least action (by varying the fields), i.e., not by using the Euler-Lagrange equations. Give an explanation of why the boundary terms do not contribute in your result, and discuss the relation to the Klein-Gordon-equation(s) you would expect for non-interacting fields.

Solution: We consider the two real scalar fields ϕ_i ($i = 1, 2$) as independent. Under a change of one of them, say $\phi_1 \rightarrow \phi_1 + \delta\phi_1$, the Lagrangian changes as $\mathcal{L} \rightarrow \mathcal{L} + \delta\mathcal{L}$ with

$$\delta\mathcal{L} = (\partial_\mu \phi_1)(\partial^\mu \delta\phi_1) - m_1^2 \phi_1 \delta\phi_1 - 2\lambda \phi_1 \phi_2^2 \delta\phi_1,$$

where we used the fact that $\delta(\partial_\mu \phi_1) \equiv \partial_\mu(\phi_1 + \delta\phi_1) - \partial_\mu \phi_1 = \partial_\mu(\delta\phi_1)$. The first term can be rewritten to $\partial^\mu[(\partial_\mu \phi_1)\delta\phi_1] - \square\phi_1(\delta\phi_1)$. After substitution into the action integral $\delta S = \int d^4x \delta\mathcal{L}$, the term $\partial^\mu[(\partial_\mu \phi_1)\delta\phi_1]$ leads to a surface integral of $\partial_\mu \phi_1(\delta\phi_1)$, by the divergence theorem in four dimensions. Thus if we restrict ourselves to variations $\delta\phi_i$ that vanish at the boundaries, we can ignore the term.

Finally, we find that

$$\delta\mathcal{L} = -[\square\phi_1 + m_1^2 \phi_1 + 2\lambda \phi_1 \phi_2^2]\delta\phi_1.$$

Requiring it to vanish for an arbitrary variation $\delta\phi_1$ leads to the equation of motion (EOM) for ϕ_1 :

$$\square\phi_1 + m_1^2 \phi_1 + 2\lambda \phi_1 \phi_2^2 = 0.$$

As the Lagrangian is symmetric under the interchange of field labels $1 \leftrightarrow 2$, the EOM for ϕ_2 immediately follows:

$$\square\phi_2 + m_2^2 \phi_2 + 2\lambda \phi_1^2 \phi_2 = 0.$$

The Lagrangian term $-\lambda \phi_1^2 \phi_2^2$ couples the two fields, leading to ϕ_2 showing up in the EOM for ϕ_1 and vice versa, meaning that they interact. Klein-Gordon equations for both fields are retrieved in the limit where the coupling vanishes ($\lambda \rightarrow 0$).

Problem 2 Obtaining ladder operators from fields

(J. Skaar)

a) The hermitian Klein-Gordon field $\phi(x)$ and its canonical momentum density $\pi(x)$ are

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right), \quad (1a)$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{-ipx} - a_{\mathbf{p}}^\dagger e^{ipx} \right), \quad (1b)$$

where $p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. Find a method to invert these relations; thus obtaining the ladder operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ from $\phi(x)$ and $\pi(x)$.

Hint: Set $t = x^0 = 0$, and try to write the expressions as Fourier transforms by changing integration variable in one of the terms for each field.

Solution: We set $t = x^0 = 0$, change integration variable $\mathbf{p} \mapsto -\mathbf{p}$ in the last terms, and use $E_{-\mathbf{p}} = E_{\mathbf{p}}$:

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger \right) e^{i\mathbf{p} \cdot \mathbf{x}}, \quad (2a)$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger \right) e^{i\mathbf{p} \cdot \mathbf{x}}. \quad (2b)$$

If $\phi(\mathbf{p})$ and $\pi(\mathbf{p})$ are the Fourier transforms of $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$, respectively, (2) shows that

$$\phi(\mathbf{p}) = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger \right), \quad (3a)$$

$$\pi(\mathbf{p}) = -i \sqrt{\frac{E_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger \right), \quad (3b)$$

or

$$a_{\mathbf{p}} = \sqrt{\frac{E_{\mathbf{p}}}{2}} \phi(\mathbf{p}) + \frac{i}{\sqrt{2E_{\mathbf{p}}}} \pi(\mathbf{p}), \quad (4a)$$

$$a_{\mathbf{p}}^\dagger = \sqrt{\frac{E_{\mathbf{p}}}{2}} \phi^\dagger(\mathbf{p}) - \frac{i}{\sqrt{2E_{\mathbf{p}}}} \pi^\dagger(\mathbf{p}). \quad (4b)$$

Thus we can find the ladder operators from the fields by setting $x^0 = 0$, Fourier transform them, and plugging into (4). (Note that even though $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ are hermitian (corresponding to real, classical fields), the Fourier transformed fields $\phi(\mathbf{p})$ and $\pi(\mathbf{p})$ are not necessarily hermitian.)

b) By requiring equal-time commutators $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$, show that we must have

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}), \quad (5a)$$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0. \quad (5b)$$

Hint: Follows from the result in a), after determining $[\phi(\mathbf{p}), \pi^\dagger(\mathbf{q})]$ and $[\phi(\mathbf{p}), \pi(\mathbf{q})]$ (the commutator of the Fourier transformed fields).

c) Prove that

$$a_{\mathbf{p}} = i \int d^3x \Psi_{\mathbf{p}}^*(x) \overleftrightarrow{\partial}_0 \phi(x) \quad (6)$$

for $x^0 = 0$. Here we have defined

$$\Psi_{\mathbf{p}}(x) = \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{-ipx}, \quad p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \quad (7)$$

and the symbol $\overleftrightarrow{\partial}_0$, which operates “in both directions”:

$$f \overleftrightarrow{\partial}_0 g = f \partial_0 g - (\partial_0 f) g. \quad (8)$$

We will use expression (6) later in the course, when discussing the important LSZ formula.

Hint: Substitute (7) and compare to the result in a).

Problem 3 The complex Klein-Gordon field (Peskin & Schroeder exercise 2.2, modified)

This problem considers the quantization of a *complex*, scalar field. It is both a good recap on how we quantized the real scalar field, and at the same time needs concepts that we meet when quantizing fermions (in particular the necessary appearance of anti-particles). The Lagrangian is

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi.$$

It is easiest to analyze this theory by considering ϕ and ϕ^* (rather than the real and imaginary parts) as the independent, dynamical variables.

a) Show that the Hamilton operator is given by

$$H = \int d^3x (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi)$$

Solution: Starting with the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi = \partial_0 \phi^* \partial^0 \phi - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi, \quad (9)$$

we find the conjugate momenta:

$$\begin{aligned} \pi &= \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} = \partial_0 \phi^* \\ \pi^* &= \frac{\partial \mathcal{L}}{\partial \partial_0 \phi^*} = \partial_0 \phi \end{aligned} \quad (10)$$

From these expressions we can calculate the Hamiltonian density:

$$\begin{aligned} \mathcal{H} &= \Sigma \pi_i \dot{\phi}_i - \mathcal{L} \\ &= \pi \pi^* + \pi^* \pi - (\pi \pi^* - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi) \\ &= \pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi, \end{aligned} \quad (11)$$

and thus the Hamiltonian is $H = \int d^3 \mathbf{x} (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi)$.

b) Compared to the real case, the theory exhibits an additional symmetry, $\phi \mapsto e^{-i\alpha} \phi$. Show that this is indeed a symmetry, and use Noether's theorem to show that the corresponding conserved charge is proportional to

$$Q = -i \int d^3 x (\dot{\phi}^* \phi - \phi^* \dot{\phi}). \quad (12)$$

Hint: A similar calculation has already been done in the lectures.

So far we have considered the classical fields ϕ and ϕ^* . We now quantize the theory, by promoting the fields to operators ϕ and $\phi^\dagger$, respectively. Similarly, the canonical momentum densities are π and $\pi^\dagger$. Complex conjugation (*) in the classical expressions above must therefore be replaced by hermitian conjugate ($\dagger$).

We require

$$\begin{aligned} [\phi(\mathbf{x}), \pi(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \\ [\phi^\dagger(\mathbf{x}), \pi^\dagger(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (13)$$

while

$$\begin{aligned} [\phi(\mathbf{x}), \phi(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \pi(\mathbf{y})], \\ [\phi(\mathbf{x}), \phi^\dagger(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \pi^\dagger(\mathbf{y})], \\ [\phi(\mathbf{x}), \pi^\dagger(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \phi^\dagger(\mathbf{y})]. \end{aligned} \quad (14)$$

c) In the Heisenberg picture the fields evolve according to the Heisenberg equations of motion,

$$i \frac{\partial \phi(x)}{\partial t} = [\phi(x), H] \quad (15a)$$

$$i \frac{\partial \pi(x)}{\partial t} = [\pi(x), H]. \quad (15b)$$

Prove that ϕ and π satisfy the following Hamilton's equations of motion:

$$\frac{\partial \phi(x)}{\partial t} = \pi^\dagger(x), \quad (16a)$$

$$\frac{\partial \pi(x)}{\partial t} = (\nabla^2 - m^2)\phi^\dagger(x). \quad (16b)$$

From these deduce that ϕ satisfies the Klein-Gordon equation.

Solution:

$$\begin{aligned} i\frac{\partial \phi(x)}{\partial t} &= [\phi(x), H] = \left[\phi(x), \int d^3\mathbf{y} (\pi^\dagger \pi + \nabla \phi^\dagger \cdot \nabla \phi + m^2 \phi^\dagger \phi) \right] \\ &= \int d^3\mathbf{y} [\phi(x), \pi(y)] \pi^\dagger(y) \\ &= \int d^3\mathbf{y} (i\delta^{(3)}(\mathbf{x} - \mathbf{y})) \pi^\dagger(y) \\ &= i\pi^\dagger(x). \end{aligned} \quad (17)$$

Here we used the fact that ϕ commute with all terms in $\mathcal{H}$, except π . To derive the next Hamilton equation, it is useful to note that

$$H = \int d^3\mathbf{y} (\pi^\dagger \pi + \nabla(\phi \nabla \phi^\dagger) - \phi \nabla^2 \phi^\dagger + m^2 \phi^\dagger \phi). \quad (18)$$

Here the second term can be written as a surface integral of $\phi \nabla \phi^\dagger$. After substitution into

$$i\frac{\partial \pi(x)}{\partial t} = [\pi(x), H] \quad (19)$$

the first two terms of H can be ignored; the first because π commutes with $\pi^\dagger \pi$, and the second because we may take the surface integral sufficiently far out, away from $\mathbf{x}$, which ensures that the commutator vanishes. The last two terms lead to terms proportional to a delta function, which after integration give

$$i\frac{\partial \pi(x)}{\partial t} = i(\nabla^2 \phi^\dagger - m^2 \phi^\dagger). \quad (20)$$

By combining (17) and (20) we obtain the Klein-Gordon equation of motion:

$$\frac{\partial^2 \phi(x)}{dt^2} - \nabla^2 \phi(x) + m^2 \phi(x) = 0. \quad (21)$$

d) Introduce annihilation and creation operators, and show that

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}) \quad (22)$$

if we ignore an infinite constant (as before). Here you should convince yourself that the Fourier decomposition of a complex function $\phi(\mathbf{x})$, already at the classical level, requires twice as many independent Fourier coefficients as for a real function. When promoted to operators, you should thus denote those with, e.g., $a_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$ (rather than $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$).

Argue that the theory contains two types of particles with mass m .

Solution: We start by introducing creation and annihilation operators for the field ϕ , noting now that ϕ is no longer hermitian ($\phi \neq \phi^\dagger$). Therefore, we cannot assume that the coefficient of $e^{ip \cdot x}$ is just the adjoint of that of $e^{-ip \cdot x}$. We are thus forced to introduce two sets of operators, $a_{\mathbf{p}}$ and $b_{\mathbf{p}}^\dagger$:

$$\begin{aligned}\phi(x) &= \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ip \cdot x} + b_{\mathbf{p}}^\dagger e^{ip \cdot x}) \Big|_{p^0=E_{\mathbf{p}}} \\ \phi^\dagger(x) &= \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}}^\dagger e^{ip \cdot x} + b_{\mathbf{p}} e^{-ip \cdot x}) \Big|_{p^0=E_{\mathbf{p}}} \\ \pi(x) = \dot{\phi}(x) &= \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (iE_{\mathbf{p}} a_{\mathbf{p}}^\dagger e^{ip \cdot x} - iE_{\mathbf{p}} b_{\mathbf{p}} e^{-ip \cdot x}) \Big|_{p^0=E_{\mathbf{p}}} \\ \pi^\dagger(x) = \dot{\phi}^\dagger(x) &= \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (-iE_{\mathbf{p}} a_{\mathbf{p}} e^{-ip \cdot x} + iE_{\mathbf{p}} b_{\mathbf{p}}^\dagger e^{ip \cdot x}) \Big|_{p^0=E_{\mathbf{p}}}\end{aligned}\quad (23)$$

The Hamiltonian must be independent of time (why?), so we set $x^0 = 0$. Plugging into the Hamiltonian,

$$\begin{aligned}H &= \int d^3\mathbf{x} (\pi^\dagger \pi + \nabla \phi^\dagger \cdot \nabla \phi + m^2 \phi^\dagger \phi) \\ &= \int d^3\mathbf{x} \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (-iE_{\mathbf{p}} a_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} + iE_{\mathbf{p}} b_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}}) \\ &\quad \cdot \int \frac{d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} (iE_{\mathbf{q}} a_{\mathbf{q}}^\dagger e^{-i\mathbf{q} \cdot \mathbf{x}} - iE_{\mathbf{q}} b_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{x}}) \\ &\quad + \int d^3\mathbf{x} \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (i\mathbf{p} a_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}} - i\mathbf{p} b_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}}) \\ &\quad \cdot \int \frac{d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} (-i\mathbf{q} a_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{x}} + i\mathbf{q} b_{\mathbf{q}}^\dagger e^{-i\mathbf{q} \cdot \mathbf{x}}) \\ &\quad + m^2 \int d^3\mathbf{x} \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}} + b_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}}) \\ &\quad \cdot \int \frac{d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{x}} + b_{\mathbf{q}}^\dagger e^{-i\mathbf{q} \cdot \mathbf{x}}).\end{aligned}\quad (24)$$

Performing the $\int d^3\mathbf{x}$ integral yields a delta function, which collapses the integration with respect to $\mathbf{q}$. After a little work this leads to

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}} b_{\mathbf{p}}^\dagger) = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}). \quad (25)$$

For the last equality, we dropped an infinite constant. In such calculations it is also useful to note that $E_{-\mathbf{p}} = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, and it may be helpful to use the substitution $-\mathbf{p} \mapsto \mathbf{p}$ in some integrals.

The final expression is exactly as for the real (or hermitian) Klein-Gordon field, except that now there are two (number operator) terms $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ and $b_{\mathbf{p}}^\dagger b_{\mathbf{p}}$. As observables these count the number of “a” and “b” particles in mode $\mathbf{p}$.

From (13) and (14) it is possible to obtain the usual ladder operator commutators for $a_{\mathbf{p}}$, and for $b_{\mathbf{p}}$, and that all commutators between

$a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ operators vanish. Thus we must consider the “a” and “b” excitations as different particles.

e) Rewrite the conserved charge (12) in terms of the annihilation and creation operators introduced above. How does the charge between the two types of particles differ?

Solution: Starting from the expression found in b), plugging in field expressions and using commutation relations, we find after some work

$$Q = \int \frac{d^3p}{(2\pi)^3} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - b_{\mathbf{p}}^\dagger b_{\mathbf{p}}). \quad (26)$$

Note how particles created by the $a^\dagger$ and $b^\dagger$ operators get opposite charge!