University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 4

Problem 1 Single particles in wave packets (J. Skaar)

In this problem, you will construct single particles in wave packets and view their propagation in space. The nonrelativistic and ultrarelativistic limits will be examined. A final goal is to understand the definition of the S-matrix.

Note that usually p^2 means $p^2 = (p^0)^2 - \mathbf{p}^2$, but when we limit ourselves to one spatial dimension, p may refer to one-dimensional momentum.

a) Our particle will be expressed as

$$|\psi\rangle = a_\psi^\dagger |0\rangle\,, \quad a_\psi^\dagger = \int \frac{d^3p}{(2\pi)^3}\psi(\mathbf{p})a_\mathbf{p}^\dagger, \tag{1}$$

where $a_\mathbf{p}^\dagger$ is the usual creation operator for the mode with momentum $\mathbf{p}$. Here $a_\psi^\dagger$ should be viewed as a wavepacket creation operator. We let the wavepacket spectrum $\psi(\mathbf{p})$ be normalized by

$$\int \frac{d^3p}{(2\pi)^3}|\psi(\mathbf{p})|^2 = 1.$$

Calculate $[a_\psi, a_\psi^\dagger]$, $\langle\psi|\psi\rangle$, and $[a_\mathbf{p}, a_\psi^\dagger]$.

Solution:

$$\begin{aligned}[a_\psi, a_\psi^\dagger] &= \int \frac{d^3p}{(2\pi)^3}\psi^*(\mathbf{p}) \int \frac{d^3q}{(2\pi)^3}\psi(\mathbf{q})\left[a_\mathbf{p}, a_\mathbf{q}^\dagger\right] \\ &= \int \frac{d^3p}{(2\pi)^3}|\psi(\mathbf{p})|^2 = 1,\end{aligned}$$

$$\langle\psi|\psi\rangle = \langle 0|\, a_\psi a_\psi^\dagger\, |0\rangle = \langle 0|\left(1 + a_\psi^\dagger a_\psi\right)|0\rangle = 1,$$

and

$$[a_\mathbf{p}, a_\psi^\dagger] = \psi(\mathbf{p}).$$

b) One observable is the field operator itself,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}e^{-ipx} + a_\mathbf{p}^\dagger e^{ipx}\right), \quad p^0 = E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}. \tag{2}$$

Show that $\langle\psi|\,\phi(x)\,|\psi\rangle = 0$. This does not mean that a measurement of the field in this state always gives zero; it may fluctuate around zero with zero expectation value. Indeed, prove that the expectation value of the normal-ordered ($\mathcal{N}$) observable $\phi^2(x)$ is

$$\langle\psi|\,\mathcal{N}\{\phi^2(x)\}\,|\psi\rangle = 2|\Psi(x)|^2\,,$$

where

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{\psi(\mathbf{p})}{\sqrt{2E_\mathbf{p}}} e^{-ipx}. \tag{3}$$

Normal ordering means to put all creation operators to the left, i.e., $\mathcal{N}(aa^\dagger) = a^\dagger a$ and $\mathcal{N}(a^\dagger a) = a^\dagger a$.

Note: The free field operator (2) *only contains a standard, plane-wave x-dependence. Thus, although the Heisenberg picture state $|\psi\rangle$ is independent of space and time, the actual space and time dependence of the particle is encoded into $|\psi\rangle$!*

Hint: Substitute the field expression in $\langle 0| a_\psi \mathcal{N}\{\phi^2(x)\} a_\psi^\dagger |0\rangle$. Use the commutator $[a_\mathbf{p}, a_\psi^\dagger] = \psi(\mathbf{p})$.

c) We will now consider the expectation value $|\Psi(x)|^2$ in more detail, thinking of it as the "shape of the particle". Argue that $\Psi(\mathbf{x}, t = 0)$ can have any shape in space, by tailoring the wavepacket spectrum $\psi(\mathbf{p})$. Choose $\psi(\mathbf{p})/\sqrt{2E_\mathbf{p}}$ such that $|\Psi(\mathbf{x}, t = 0)|^2$ is a 3d gaussian centered at the point $\mathbf{b}$.

Solution: For $t = 0$, the definition of $\Psi(x)$ is a 3d Fourier transform. Thus by tailoring the spectrum, we can get any $\Psi(\mathbf{x}, t = 0)$. We have that

$$\frac{\psi(\mathbf{p})}{\sqrt{2E_\mathbf{p}}} \propto e^{-\frac{(\mathbf{p}-\mathbf{p}_0)^2}{2(\Delta p)^2}} e^{-i\mathbf{b}\cdot\mathbf{p}}$$

leads to a 3d gaussian about $\mathbf{b}$ (for any $\mathbf{p}_0$), as seen from the shift property of the Fourier transform (and the fact that the Fourier transform of a gaussian is a gaussian).

d) Peskin & Schroeder (p. 24) claims that the state

$$\phi(\mathbf{b})\,|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-i\mathbf{p}\cdot\mathbf{b}}}{\sqrt{2E_\mathbf{p}}} a_\mathbf{p}^\dagger\,|0\rangle$$

describes a particle at location $\mathbf{b}$. (See (2.41) and (2.35).) Here $\phi(\mathbf{b})$ denotes the field operator at position $\mathbf{x} = \mathbf{b}$ and a fixed time $t = 0$. Note that this state can be put in the form (1). Find $\psi(\mathbf{p})/\sqrt{2E_\mathbf{p}}$ for this state, and show that $\Psi(\mathbf{x}, t = 0)$ can be expressed in the form of a Fourier integral. Argue that the state $\phi(\mathbf{b})\,|0\rangle$ is not perfectly localized to a point at $\mathbf{x} = \mathbf{b}$.

Solution: Comparison gives

$$\frac{\psi(\mathbf{p})}{\sqrt{2E_\mathbf{p}}} = \frac{e^{-i\mathbf{p}\cdot\mathbf{b}}}{2E_\mathbf{p}}$$

and

$$\Psi(\mathbf{x}, t = 0) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-i\mathbf{p}\cdot\mathbf{b}}}{2E_\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}.$$

This is a rather simple Fourier integral. If $E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$ were constant (which is the case in the nonrelativistic limit $|\mathbf{p}| \ll m$), we

would obtain a result $\propto \delta(\mathbf{x}-\mathbf{b})$. However, $E_{\mathbf{p}}$ is not constant, so the Fourier transform will not be a delta function. It is possible to calculate the integral, and then one finds that it has, in fact, infinite tails.

It is natural to suggest that we should rather pick $\psi(\mathbf{p})/\sqrt{2E_{\mathbf{p}}} = e^{-i\mathbf{p}\cdot\mathbf{b}}$, which would lead to $\Psi(\mathbf{x},\, t=0) \propto \delta(\mathbf{x}-\mathbf{b})$. However, one may then show that the expectation value of other observables, e.g. $\langle\psi|\mathcal{N}\{(\partial_0\phi)^2(x)\}|\psi\rangle$, are not perfectly localized. In fact, one can show that the single particle cannot be perfectly localized no matter how we pick the wavepacket spectrum.

e) Define a position wave function as in nonrelativistic quantum mechanics: $\psi(x) \equiv \langle x|\psi\rangle$, where $|x\rangle = \phi(x)\,|0\rangle$. Show that $\psi(x)$ is the same function as $\Psi(x)$ in (3).

Hint:

$$\psi(x) = \langle x|\psi\rangle = \langle 0|\,\phi(x)a^\dagger_\psi\,|0\rangle\,.$$

Substitute the expression for $\phi(x)$ and note that $[a_{\mathbf{p}}, a^\dagger_\psi] = \psi(\mathbf{p})$.

f) Prove that the wave function $\Psi(x)$ in (3) obeys the Schrödinger equation in the nonrelativistic limit $|\mathbf{p}| \ll m$. (More precisely, we require $|\mathbf{p}| \ll m$ for all $\mathbf{p}$'s contributing to the integral in (3).)

Hint: It is tempting to approximate $E_{\mathbf{p}} \approx m$ in this limit, but then we get a trivial time dependence e^{-imt}, which gives no time dependence for $|\Psi(x)|^2$. So we need to go to the next order:

$$E_{\mathbf{p}} = m\sqrt{1+\mathbf{p}^2/m^2} \approx m + \mathbf{p}^2/2m$$

Then substitute in (3), ignore the trivial time dependence due to m, and verify that (3) satisfies the Schrödringer equation.

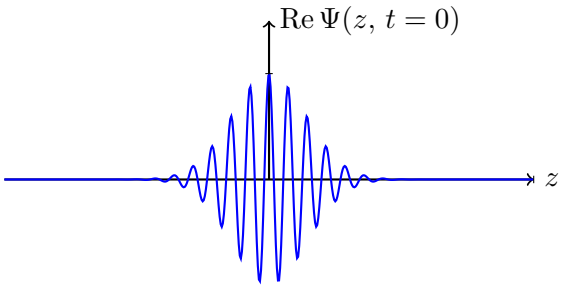
g) Consider a particle with narrow wavepacket spectrum about some central value $\mathbf{p}_0$. Let $\mathbf{p}_0$ point in the z-direction, and assume that the spectrum $\Psi(x)$ only contains values for $\mathbf{p}$ along the z-direction. In other words, consider the 1+1d case

$$\Psi(z,t) = \int \frac{dp}{2\pi}\frac{\psi(p)}{\sqrt{2E}}e^{-iEt+ipz}, \quad E = \sqrt{p^2+m^2}.$$

If $\psi(p)/\sqrt{2E}$ is a narrow gaussian centered about $p = p_0$, make a rough sketch of $\mathrm{Re}\,\Psi(z,\, t=0)$. Indicate how the width of the pulse is related to the width of the gaussian spectrum $\psi(p)$.

Solution: The Fourier transform of a gaussian is a gaussian, and the two widths are inversally proportional according to the general Fourier transform property. The width σ of the pulse in the figure is $\sigma \sim 1/\Delta p$, where Δp is the momentum bandwidth of the wavepacket, i.e., the width

of $\psi(p)$. The oscillations has wavelength $\sim 1/p_0$. Since the spectrum is narrow ($\Delta p \ll p_0$), we have $1/p_0 \ll 1/\Delta p$ so there are several oscillations in the pulse.



h) Still in 1+1d, we now consider the propagation of the wavepacket in the ultrarelativistic limit $p \gg m$. Prove that in this case the pulse moves to the right (in the $+z$-direction), with no dispersion/distortion.

Solution: In this limit we may identify $E = p$, and rewrite the exponent such that the z, t dependence is in the form $z - t$. Thus the wave packet propagates at the light velocity without dispersion.

i) We now consider the propagation of $\Psi(z,t)$ in the nonrelativistic limit $p \ll m$. As before, we do not care what the overall constant is, we are only interested in the overall shape and width of $|\Psi(z,t)|^2$. Let $\psi(p)/\sqrt{2E}$ be a narrow gaussian, and use your favorite methods (by hand, or using an algebraic or numeric computer program) to obtain rough plots of $|\Psi(z,t)|^2$ as a function of z for a $t \ll m\sigma^2$ and a $t \gg m\sigma^2$. Or, since this is a little bit boring, perhaps just search the Internet for something like "gaussian wave packet in Schrödinger equation". For example there is a nice Wikipedia page "wave packet".

What happens with the pulse as the time goes by? What is the time $t \approx m\sigma^2$ in SI units?

Solution: We have

$$\Psi(z,t) = e^{-imt} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{(p-p_0)^2}{2(\Delta p)^2}} e^{-i\frac{p^2}{2m}t + ipz},$$

which can be put into the form $\int du \exp(-au^2 + bu)$ and solved easily in principle. However, we prefer to search the Internet and find out that the pulse broaden as the time goes by. Converting the time $t = m\sigma^2$ to SI units amounts to multiplying by a suitable factor of $\hbar^n c^m$ to obtain the right dimension. This gives $t = m\sigma^2/\hbar$. We find that the pulse starts to broaden earlier if the pulse is short. The reason is that short pulses have broad spectra, giving more dispersion for the frequency components.

j) Consider (3) in the 2+1d case

$$\Psi(y,z) = \int \frac{dp_y dp_z}{(2\pi)^2} \frac{\psi(\mathbf{p})}{\sqrt{2E_{\mathbf{p}}}} e^{-iE_{\mathbf{p}}t + ip_y y + ip_z z}. \tag{4}$$

Let $\psi(\mathbf{p})/\sqrt{2E_{\mathbf{p}}} = \delta(p_z - p_0)e^{-\frac{p_y^2}{2(\Delta p)^2}}$. How does the wave envelope $|\Psi(y,z)|^2$ look like for $t = 0$? What happens if we include a factor $e^{-ip_y b}$ in the expression for $\psi(\mathbf{p})/\sqrt{2E_{\mathbf{p}}}$?

Solution: We find for $t = 0$ that $\Psi(y,z)$ is $e^{ip_0 z}$ times the Fourier transform of a gaussian of width Δp along p_y, which is a new gaussian of width $\sigma = 1/\Delta p$ along y. Thus this is a beam along the z-axis, of thickness σ. (Draw a figure.) If we include a factor $e^{-ip_y b}$, the shift property of the Fourier transform shows that the beam gets centered about $y = b$ rather than $y = 0$.

k) In scattering experiments, which are quite central in QFT, two particles are initially far apart, but overlap after some time. We consider 2+1d or 3+1d. Why do you think we need to assume that the particles are contained in wavepackets? In other words, why can't we just assume that the particles each have a single momentum, p_1 and p_2?

Solution: Single momentum particles (plane waves) have infinite extent, and overlap at any time. To build a finite beam, we need a spectrum of wavevectors/momenta.

l) Still, in scattering experiments the wavepackets have usually narrow momentum spectra such that they are much like plane waves. More precisely, $|\Delta\mathbf{p}| \ll |\mathbf{p}_0|$, where $\Delta\mathbf{p}$ is the variation in momentum. This means that the size of the pulse is much larger than a de Broglie wavelength, in all directions. Even though we cannot use true plane waves in the analysis, we can come as close as we wish to plane waves, by setting up the in states in sufficiently remote past, sufficiently far away.

In Peskin & Schroeder p. 102-104, the S-matrix is introduced as the overlap between so-called in and out states. Read p. 102-103 and the first paragraph on p. 104. Try to understand what is meant by the in and out states (and (4.70)). Consider the special case with a free Klein-Gordon theory (which is the one treated in this exercise). What is the S-matrix in this case? Relate to this exercise.

Hint: When there is no interaction, the set of in states and out states are identical. Thus the S-matrix is the identity. When there is interaction, the in states are like free almost-plane-waves in the remote past, while the out states are like free almost-plane-waves in the remote future. However, they are Heisenberg states, with a common reference time for the Heisenberg picture.