

FYS4170 Relativistic quantum field theory

Welcome to Relativistic quantum field theory!

Course information on semester page.

QFT challenging, but also extremely interesting.
Together with GR it is the most fundamental theory.

It is the basis for, or quite central for

- particle physics
- advanced solid state physics
- quantum optics
- quantum information technology or theory

Why QFT?

- Nonrelativistic QM is noncausal.
- QM cannot describe the generation or annihilation of particles (high energy physics).
- QM does not provide a proper description of photons (low energy, everyday physics).
(We don't need full QFT for low energy quantum optics,
but the photon part must be treated as a quantum field.)

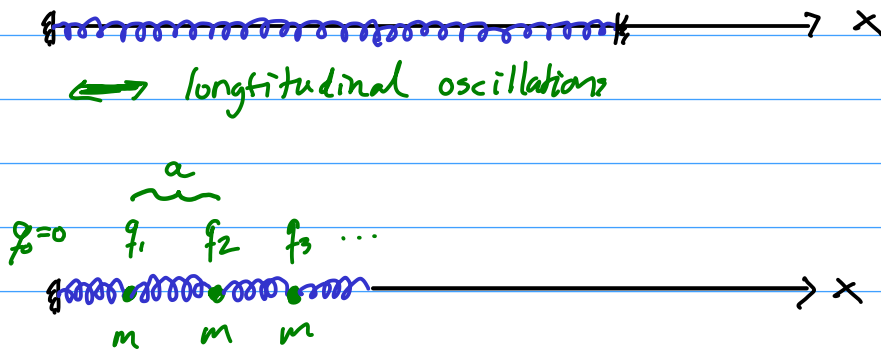
Classical field theory

In classical mechanics: $S = \int L(q_i, \dot{q}_i) dt$

$\dot{q}_1$ $\dot{q}_2$ $\dot{q}_3$

Eqs. of motion obtained $\delta S = 0$ (action stationary)
(Hamilton's principle)

Ex: String



$$L = T - V = \sum_i \frac{1}{2} m \dot{q}_i^2 - \sum_i \frac{1}{2} k (q_{i+1} - q_i)^2$$

$$= \sum_i a \left[\frac{1}{2} \frac{m}{a} \dot{q}_i^2 - \frac{1}{2} k a \left(\frac{q_{i+1} - q_i}{a} \right)^2 \right]$$

Limit $a \rightarrow 0$, $q_i(t) \rightarrow q(x, t)$

$$\rightarrow \int dx \left[\frac{1}{2} \mu \left(\frac{\partial q}{\partial t} \right)^2 - \frac{1}{2} \frac{(ka)}{Y} \left(\frac{\partial q}{\partial x} \right)^2 \right]$$

$$\left. \begin{array}{l} \text{spring with } k \\ \text{spring with } 2k \end{array} \right\} \begin{array}{l} F = k q \\ F = 2k q \end{array} \Rightarrow Y = ka = \text{const.}$$

$$S = \int L dt = \iint dx dt \underbrace{\left[\frac{1}{2} \mu \dot{q}^2 - \frac{1}{2} Y \left(\frac{\partial q}{\partial x} \right)^2 \right]}_L$$

Lagrangian field theory

$$S = \int L dt = \int \mathcal{L}(\varphi, \partial_\mu \varphi) d^4x, \quad \mathcal{L} = \text{Lagrange density, Lagrangian.}$$

Principle of least action:

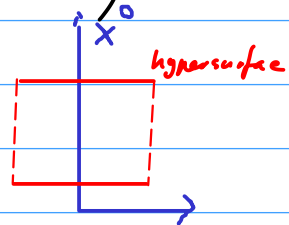
$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta (\partial_\mu \varphi) \right\}$$

$$= \int d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) \delta \varphi \right] + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi \right) \right\}$$

since $\partial_\mu (\delta \varphi) = \delta \partial_\mu \varphi$.

Recall $\int d^3x \nabla \cdot \vec{A} = \int d\vec{S} \cdot \vec{A}$ (divergence theorem)

$$\int d^4x \partial_\mu K^\mu = \text{surface integral of } \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi$$



Restrict ourselves to (any) variations $\delta \varphi$ such that $\delta \varphi = 0$ on the surface.

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} = 0 \quad \text{Euler-Lagrange}$$

Ex: String again. "Field": $\varphi = q(x, t)$

$$\mathcal{L} = \frac{1}{2} \mu \left(\frac{\partial q}{\partial t} \right)^2 - \frac{1}{2} Y \left(\frac{\partial q}{\partial x} \right)^2$$

Note that x corresponds to the index in the discrete, class. mech. case

$$E-L: \frac{\partial \mathcal{L}}{\partial q} - \partial_t \frac{\partial \mathcal{L}}{\partial (\partial_t q)} - \partial_x \frac{\partial \mathcal{L}}{\partial (\partial_x q)} = 0$$

$$0 - \partial_t \mu \frac{\partial q}{\partial t} + \partial_x Y \partial_x \frac{\partial q}{\partial x} = 0$$

$$- \mu \frac{\partial^2 q}{\partial t^2} + Y \frac{\partial^2 q}{\partial x^2} = 0$$

(wave eq.)

Hamiltonian field theory

Ex: String. $p_i = \frac{\partial L}{\partial \dot{q}_i}$ conjugate momentum
 $= m \dot{q}_i$

$$H = \sum_i p_i \dot{q}_i - L = \sum_i m \dot{q}_i^2 - L$$

$$= \sum_i \frac{1}{2} m \dot{q}_i^2 + \frac{1}{2} k (q_{i+1} - q_i)^2$$

Cont. limit: $H = \int dx \left[\frac{1}{2} \mu \left(\frac{\partial \varphi}{\partial t} \right)^2 + \frac{1}{2} Y \left(\frac{\partial \varphi}{\partial x} \right)^2 \right]$

Define conjugate momentum density $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$

$$H = \int d^3x [\pi \dot{\varphi} - \mathcal{L}] = \int d^3x \mathcal{H}$$

$\begin{aligned} \mathcal{H} &= \pi \dot{\varphi} - \mathcal{L} \\ &= \dot{\varphi}^2 - \mathcal{L} \\ &= \frac{1}{2} \mu (\dot{\varphi})^2 + \frac{1}{2} Y (\varphi_x')^2 \\ &\text{since } \pi = \dot{\varphi}. \end{aligned}$

Ex: Important example Klein-Gordon field

$$\mathcal{L} = \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{2} m^2 \varphi^2 = \frac{1}{2} (\partial_\mu \varphi) (\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2$$

(same as string except last term) PS: $(\partial_\mu \varphi)^2 = (\partial_\mu \varphi) (\partial^\mu \varphi)$

E-L: $\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} = 0$

$$-m^2 \varphi - \partial_\mu \partial^\mu \varphi = 0$$

$$[\partial_\mu \partial^\mu + m^2] \varphi = 0, \text{ Klein-Gordon-eq.}$$

$$= g_{\mu\beta} (\delta_\mu^\alpha x^\beta + x^\alpha \delta_\mu^\beta) [\partial_\alpha^2 - \nabla^2 + m^2] \varphi = 0$$

$$= x_\mu + x_\mu$$

$$= 2x_\mu$$

$$\partial^\mu x^2 = 2x^\mu$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}$$

$$\mathcal{H} = \pi \dot{\varphi} - \mathcal{L} = \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} m^2 \varphi^2$$

Noether's theorem

Continuous symmetry $\Rightarrow$ conservation law

Ex: translational symmetry $\Rightarrow$ conserved momentum
time - $\rightarrow$ $\Rightarrow$ conserved energy

Transforming the field: $\varphi(x) \mapsto \varphi'(x) = \varphi(x) + \alpha \Delta \varphi(x)$
 $\Rightarrow \mathcal{L}(x) \mapsto \mathcal{L}'(x) = \mathcal{L}(x) + \alpha \Delta \mathcal{L}(x)$
Let α be infinitesimal.

Def. symmetry: eq. of motion invariant

(1) Assume $\mathcal{L}(x) \mapsto \mathcal{L}(x) + \alpha \partial_\mu J^\mu$ for some J^μ .
Then we have a symmetry: $S = \int \mathcal{L}(x) d^4x + \text{surface t.}$

(2) An independent way of obtaining Delta L (using the eqs of motion):

$$\alpha \Delta \mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial \varphi} (\alpha \Delta \varphi) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \partial_\mu (\alpha \Delta \varphi)$$

Note: (1) must be a property of the Lagrangian. We must be able to find a J^μ independently of the method (2) when employing the eqs of motion. That is, find Delta L in two different ways.

Here we assume that L does not depend explicitly on x.

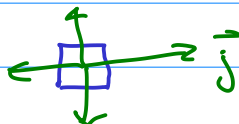
$$= \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \Delta \varphi \right) + \alpha \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) \right]}_{E-L: = 0} \Delta \varphi$$

$$\Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \Delta \varphi - J^\mu \right) = 0$$

$$\Rightarrow \partial_\mu j^\mu = 0 \quad \text{for } j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \Delta \varphi - J^\mu$$

$$\nabla \cdot \vec{j} = - \frac{\partial \rho}{\partial t}$$

$$j^\mu = (\rho, \vec{j})$$



Noether current
(not necessarily electric current).

$$Q = \int_{\text{all space } v} j^0 d^3x, \quad \frac{dQ}{dt} = \int_v \frac{\partial j^0}{\partial t} d^3x$$

$$= \int_v -\nabla \cdot \vec{j} d^3x$$

$$= \oint_S \vec{j} \cdot d\vec{s} = 0 \text{ if } v \text{ is all space}$$

Q is conserved.

If more than one field, e.g. $\phi_1, \phi_2, \dots$:

$$j^\mu = \sum_r \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \Delta \phi_r - \mathcal{J}^\mu$$

Ex: $\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m \phi \phi^*$

Consider ϕ, ϕ^* as indep. fields (could instead use $\text{Re}\phi$ and $\text{Im}\phi$)

Symmetry: $\phi \mapsto e^{i\alpha} \phi, \quad \mathcal{L} \mapsto \mathcal{L}' = \mathcal{L}$
 $(\Delta \mathcal{L} = 0, \mathcal{J}^\mu = 0)$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \Delta \phi^*$$

$$= \partial^\mu \phi^* \Delta \phi + \partial^\mu \phi \Delta \phi^*$$

$$(\partial_\mu \phi \partial^\mu \phi^* = \partial^\mu \phi \partial_\mu \phi^*)$$

$$(\Delta \phi = e^{i\alpha} \phi - \phi = (1 + i\alpha)\phi - \phi = i\alpha \phi)$$

$$= \partial^\mu \phi^* i\alpha \phi - \partial^\mu \phi i\alpha \phi^*$$

$$\frac{j^\mu}{\alpha} = i(\partial^\mu \phi^*) \phi - i\phi^* (\partial^\mu \phi)$$

This Noether current is conserved ($\partial_\mu j^\mu = 0$)

Ex: Translations: $x^\mu \mapsto x^\mu + a^\mu$

$$\begin{aligned}\varphi(x) &\mapsto \varphi(x+a) = \varphi(x) + a^\mu \partial_\mu \varphi(x) \\ \mathcal{L}(x) &\mapsto \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L}(x) \\ &= \mathcal{L}(x) + a^\mu \partial_\mu (\delta^\mu_\nu \mathcal{L})\end{aligned}$$

Assuming $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$ does not depend explicitly on x , then Noether's theorem applies.

For $a^0 = \alpha$, $a^i = 0$:

$$\begin{aligned}j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta \varphi - \delta^\mu_0 \mathcal{L} \\ &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_0 \varphi - \delta^\mu_0 \mathcal{L}\end{aligned}$$

In general: $T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L}$
energy-momentum tensor.

$$T^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \cdot \dot{\varphi} - \mathcal{L} = \pi \dot{\varphi} - \mathcal{L} = \mathcal{H}$$

$$T^{0i} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \partial^i \varphi = \pi \partial^i \varphi = -\pi \partial_i \varphi$$

$$P^i = \int T^{0i} d^3x : \text{phys. momentum carried by the field}$$

Recap: Classical field theory. Lagrangian $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$

$S = \int d^4x \mathcal{L}(x)$, $\delta S = 0 \Rightarrow$ EL-eq. / eq. of motion

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0$$

More than one field: One EL-eq. for each field (why?)

Noether's theorem: Cont., infinitesimal transformation

$$\varphi(x) \mapsto \varphi(x) + \alpha \Delta \varphi(x)$$

$$\mathcal{L}(x) \mapsto \mathcal{L}(x) + \alpha \partial_\mu J^\mu$$

$$\partial_\mu j^\mu = 0, \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta \varphi - \delta^\mu_0 \mathcal{L}$$

Quantization of the Klein-Gordon field

Classical, real KG field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}m^2 \varphi^2$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}, \quad 2\mathcal{L} = \pi \dot{\varphi} - \mathcal{L} = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\nabla \varphi)^2 + \frac{1}{2}m^2 \varphi^2$$

$$\text{KG eq.: } [\partial_\mu \partial^\mu + m^2] \varphi = [\partial_t^2 - \nabla^2 + m^2] \varphi = 0 \quad (\text{eq of motion})$$

$$\text{Fourier repr.: } \varphi(\vec{x}, t) = \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \varphi(\vec{p}, t)$$

↖ Fourier transf. of $\varphi(\vec{x}, t)$

$$\varphi(\vec{x}, t) \text{ real} \Rightarrow \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{-i\vec{p} \cdot \vec{x}} \varphi^*(\vec{p}, t) = \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \varphi(\vec{p}, t)$$

$\parallel \vec{p} \mapsto -\vec{p}$

$$\int \frac{d^3 \vec{p}}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \varphi^*(-\vec{p}, t) \quad \text{for all } \vec{x}$$

$$\Rightarrow \varphi^*(-\vec{p}, t) = \varphi(\vec{p}, t) \quad \text{for all } \vec{p}$$

$$\text{KG eq.} \Rightarrow (\partial_t^2 - (i\vec{p})^2 + m^2) \varphi(\vec{p}, t) = 0$$
$$(\partial_t^2 + \omega_{\vec{p}}^2) \varphi(\vec{p}, t) = 0, \quad \omega_{\vec{p}}^2 = \vec{p}^2 + m^2$$

Each Fourier component is a harmonic oscillator.

$$\text{Solution: } \varphi(\vec{p}, t) = \frac{1}{\sqrt{2\omega_{\vec{p}}}} [a_{\vec{p}} e^{-i\omega_{\vec{p}} t} + b_{\vec{p}}^* e^{i\omega_{\vec{p}} t}]$$

$$\left. \begin{aligned} \varphi^*(-\vec{p}, t) &= \varphi(\vec{p}, t), \quad t=0 : a_{-\vec{p}}^* + b_{-\vec{p}}^* = a_{\vec{p}} + b_{\vec{p}} \\ \dot{\varphi}^*(-\vec{p}, t) &= \dot{\varphi}(\vec{p}, t), \quad t=0 : a_{-\vec{p}}^* - b_{-\vec{p}}^* = -a_{\vec{p}} + b_{\vec{p}} \end{aligned} \right\}$$
$$\Rightarrow b_{\vec{p}} = a_{-\vec{p}}^*$$

$$\varphi(\vec{x}, t) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a_{\vec{p}} e^{-i\omega_{\vec{p}} t + i\vec{p} \cdot \vec{x}} + a_{-\vec{p}}^* e^{i\omega_{\vec{p}} t + i\vec{p} \cdot \vec{x}} \right]$$
$$= \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a_{\vec{p}} e^{-i\omega_{\vec{p}} t + i\vec{p} \cdot \vec{x}} + a_{\vec{p}}^* e^{i\omega_{\vec{p}} t - i\vec{p} \cdot \vec{x}} \right]$$

↖ $\vec{p} \mapsto -\vec{p}$ second term, $\omega_{-\vec{p}} = \omega_{\vec{p}}$.

Why discuss the (quantized) KG field?

- describes spinless particles
- simplest field that contains the main features of a quantum field!

Quantization in QM: $[q_i, p_j] = i\delta_{ij}$ ($q_i = \text{coord. part. } i$
 $p_i = \text{momentum part } i$)
 $[q_i, q_j] = [p_i, p_j] = 0$

Quantization in QFT: Fields are promoted to operators:

$$[\varphi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x}-\vec{y}) \quad [\text{PS: } i\delta^{(3)}(\vec{x}-\vec{y})]$$

$$[\varphi(\vec{x}), \varphi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0$$

Equal time commutators.

Time-dependence in Heisenberg picture:

$$\varphi(\vec{x}, t) = e^{iHt} \varphi(\vec{x}, 0) e^{-iHt}$$

$$\frac{d\varphi}{dt} = i[H, \varphi(\vec{x}, t)]$$

Instead, let's guess:

$$\varphi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p e^{-i\omega_p t + i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{i\omega_p t - i\vec{p}\cdot\vec{x}} \right]$$

$$\pi(\vec{x}, t) = \dot{\varphi}(\vec{x}, t)$$

$$= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left[a_p e^{-i\omega_p t + i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{i\omega_p t - i\vec{p}\cdot\vec{x}} \right]$$

Compact form: $\varphi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^\dagger e^{ipx} \right], \quad p^0 = \omega_p.$

Commutator rel. for $a_{\vec{p}}, a_{\vec{q}}^\dagger$: Let's guess:

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta(\vec{p}-\vec{q})$$

$$[a_{\vec{p}}, a_{\vec{q}}] = [a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0$$

Verify guess, $t=0$:

$$\begin{aligned} [\varphi(\vec{x}), \pi(\vec{y})] &= \frac{(-i)}{2} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \left(\underbrace{-[a_{\vec{p}}, a_{\vec{q}}^\dagger]}_{(2\pi)^3 \delta(\vec{p}-\vec{q})} e^{i\vec{p}\cdot\vec{x} - i\vec{q}\cdot\vec{y}} \right. \\ &\quad \left. + \underbrace{[a_{\vec{p}}^\dagger, a_{\vec{q}}]}_{-(2\pi)^3 \delta(\vec{p}-\vec{q})} e^{-i\vec{p}\cdot\vec{x} + i\vec{q}\cdot\vec{y}} \right) \end{aligned}$$

$$= \frac{(-i)}{2} \int \frac{d^3p}{(2\pi)^3} \left[-e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right]$$

$$= i \delta(\vec{x}-\vec{y}), \text{ ok!} \quad \text{Similarly: } [\varphi(\vec{x}), \varphi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})]$$

Recall:

Schr. pict.: $|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$

$A(0)$ for all t

Heis. pict.: $A(t) = e^{iHt} A(0) e^{-iHt}$

$|\psi(0)\rangle$ for all t

Expect. val.:

$\langle \psi(t) | A(0) | \psi(t) \rangle \quad \text{S.p.}$

$= \langle \psi(0) | e^{iHt} A(0) e^{-iHt} | \psi(0) \rangle$

$= \langle \psi(0) | A(t) | \psi(0) \rangle \quad \text{H.p.}$

$$\text{Hamiltonian: } H = \int d^3x \mathcal{H} \\ = \int d^3x \left[\frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} m^2 \varphi^2 \right]$$

H is independent of time (why?). Set $t=0$.

$$\varphi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}) \\ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_{\vec{p}} + a_{-\vec{p}}^\dagger) e^{i\vec{p}\cdot\vec{x}} \quad (\vec{p} \mapsto -\vec{p} \text{ in last term})$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\omega_p} (a_{\vec{p}} - a_{-\vec{p}}^\dagger) e^{i\vec{p}\cdot\vec{x}} = \dot{\varphi}(\vec{x})$$

$$H = \frac{1}{4} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \left\{ (-\omega_p \omega_q) (a_{\vec{p}} - a_{-\vec{p}}^\dagger) (a_{\vec{q}} - a_{-\vec{q}}^\dagger) e^{i(\vec{p}+\vec{q})\cdot\vec{x}} \right. \\ \left. + \frac{1}{\sqrt{\omega_p \omega_q}} (i\vec{p})(i\vec{q}) (a_{\vec{p}} + a_{-\vec{p}}^\dagger) (a_{\vec{q}} + a_{-\vec{q}}^\dagger) \cdot e^{i(\vec{p}+\vec{q})\cdot\vec{x}} \right. \\ \left. + m^2 \frac{1}{\sqrt{\omega_p \omega_q}} (a_{\vec{p}} + a_{-\vec{p}}^\dagger) (a_{\vec{q}} + a_{-\vec{q}}^\dagger) e^{i(\vec{p}+\vec{q})\cdot\vec{x}} \right\}$$

$$= \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \left\{ -\omega_p (a_{\vec{p}} - a_{-\vec{p}}^\dagger) (a_{-\vec{p}} - a_{\vec{p}}^\dagger) + \frac{1}{\omega_p} (\overbrace{\vec{p}^2 + m^2}^{\omega_p^2}) \right. \\ \left. \int e^{i(\vec{p}+\vec{q})\cdot\vec{x}} d^3x = (2\pi)^3 \delta(\vec{p}+\vec{q}) \cdot (a_{\vec{p}} + a_{-\vec{p}}^\dagger) (a_{-\vec{p}} + a_{\vec{p}}^\dagger) \right\}$$

$$= \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \omega_p \left\{ -\cancel{a_{\vec{p}} a_{-\vec{p}}} + a_{\vec{p}} a_{\vec{p}}^\dagger + a_{-\vec{p}}^\dagger a_{-\vec{p}} - \cancel{a_{-\vec{p}}^\dagger a_{\vec{p}}^\dagger} \right. \\ \left. + \cancel{a_{\vec{p}} a_{-\vec{p}}} + a_{\vec{p}} a_{\vec{p}}^\dagger + a_{-\vec{p}}^\dagger a_{-\vec{p}} + \cancel{a_{-\vec{p}}^\dagger a_{\vec{p}}^\dagger} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \frac{1}{2} (a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}}) \quad (\vec{p} \mapsto -\vec{p} \text{ last term})$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a_{\vec{p}}^\dagger a_{\vec{p}} + \underbrace{\frac{1}{2} [a_{\vec{p}}, a_{\vec{p}}^\dagger]}_{\delta(\omega)} \right)$$

$$\underline{H = \int \frac{d^3p}{(2\pi)^3} \omega_p a_{\vec{p}}^\dagger a_{\vec{p}}} \quad \uparrow \text{ ignored on the basis of ordering ambiguity}$$

Similarly:

$$\vec{P} = -\int d^3x \pi \nabla \varphi = \dots = \int \frac{d^3p}{(2\pi)^3} \vec{p} a_{\vec{p}}^\dagger a_{\vec{p}}$$

Ordering ambiguity, quantization: Harm. osc. $m=1$: $H = \frac{p^2}{2m} + \frac{1}{2} \omega^2 q^2 \mapsto \omega(a^\dagger a + \frac{1}{2})$

$$= \frac{1}{2} (\omega q - i p) (\omega q + i p) \mapsto \omega a^\dagger a$$

We have $[H, a_{\vec{p}}^{\dagger}] = \int \frac{d^3 q}{(2\pi)^3} \omega_{\vec{q}} a_{\vec{q}}^{\dagger} [a_{\vec{q}}, a_{\vec{p}}^{\dagger}] = \omega_{\vec{p}} a_{\vec{p}}^{\dagger}$
 $[H, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}}$

ψ eigenstate, energy E
 $(H a_{\vec{p}} - a_{\vec{p}} H) |\psi\rangle = -\omega_{\vec{p}} a_{\vec{p}} |\psi\rangle$
 $(H a_{\vec{p}} - a_{\vec{p}} H) |\psi\rangle = -\omega_{\vec{p}} a_{\vec{p}} |\psi\rangle$
 $H a_{\vec{p}} |\psi\rangle = (E - \omega_{\vec{p}}) a_{\vec{p}} |\psi\rangle$
 Similarly
 $H a_{\vec{p}}^{\dagger} |\psi\rangle = (E + \omega_{\vec{p}}) a_{\vec{p}}^{\dagger} |\psi\rangle$

Each mode is a harmonic oscillator.

$a_{\vec{p}}$ annihilates the vacuum: $a_{\vec{p}} |0\rangle = 0$.

$a_{\vec{p}}^{\dagger}$ creates an excitation of the harmonic osc. associated with $\vec{p}$ -mode, with energy $\omega_{\vec{p}}$: $H a_{\vec{p}}^{\dagger} |0\rangle = (a_{\vec{p}}^{\dagger} H + \omega_{\vec{p}} a_{\vec{p}}^{\dagger}) |0\rangle = \omega_{\vec{p}} a_{\vec{p}}^{\dagger} |0\rangle$
 and momentum $\vec{p}$: $\vec{P} a_{\vec{p}}^{\dagger} |0\rangle = \vec{p} a_{\vec{p}}^{\dagger} |0\rangle$

From now on, let's use $E_{\vec{p}}$ rather than $\omega_{\vec{p}}$: $E_{\vec{p}}^2 = \vec{p}^2 + m^2$
 which is the usual relativistic energy-mom. relation for particles.
 We therefore call the excitations particles!

This particle is non-localized ("plane wave particle").
 Later we will consider wave-packets.

Two particles: $a_{\vec{p}}^{\dagger} a_{\vec{q}}^{\dagger} |0\rangle = a_{\vec{q}}^{\dagger} a_{\vec{p}}^{\dagger} |0\rangle$ (not normalized)
 n particles: $(a_{\vec{p}}^{\dagger})^n |0\rangle$
 KG-particles are bosons!

Remains to verify the time-dependence of the field:

$$\varphi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} [a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^{\dagger} e^{ipx}], \quad p^0 = \omega_{\vec{p}} = E_{\vec{p}}$$

(Assumed eq. of motion KG also for the quantized field.)

We had $[H, a_{\vec{p}}] = -E_{\vec{p}} a_{\vec{p}}$.

Heisenberg eq. of motion: $i \frac{\partial A}{\partial t} = [A, H]$ for operators A .

$$\left. \begin{aligned} i \frac{\partial \varphi}{\partial t} &= i \int \frac{d^3 p}{(2\pi)^3} \frac{E_{\vec{p}}}{\sqrt{2E_{\vec{p}}}} [-i a_{\vec{p}} e^{-ipx} + i a_{\vec{p}}^{\dagger} e^{ipx}] \\ [\varphi, H] &= \int \frac{d^3 p}{(2\pi)^3} \frac{E_{\vec{p}}}{\sqrt{2E_{\vec{p}}}} [a_{\vec{p}} e^{-ipx} - a_{\vec{p}}^{\dagger} e^{ipx}] \end{aligned} \right\} \Rightarrow i \frac{\partial \varphi}{\partial t} = [\varphi, H] \quad \text{OK!}$$

Lorentz-transformation

$$(x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu)$$

Light velocity same in all frames:

$$ds^2 = dx^{\alpha 2} - dt^2 = \text{invariant}$$

$$\begin{aligned} \Rightarrow g_{\mu\nu} dx^\mu dx^\nu &= g_{\rho\sigma} dx'^\rho dx'^\sigma \\ &= g_{\rho\sigma} \Lambda^\rho_\mu dx^\mu \Lambda^\sigma_\nu dx^\nu \\ \text{! } x'^\mu &= \Lambda^\mu_\nu x^\nu \end{aligned}$$

(Λ is the same as L in Leinaas.)

$$\Rightarrow \boxed{g_{\mu\nu} = g_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu}$$

$$\begin{aligned} g_{\mu\nu} &= g_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu \\ g &= \Lambda^T g \Lambda \\ \det g &= (\det \Lambda)^2 \det g \\ \det \Lambda &= \pm 1. \end{aligned}$$

• Discrete transformations:

$$\Lambda = P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Lambda = T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

• Proper Lorentz-transf.: $\det \Lambda = +1$
 $\Lambda^0_0 > 0$

Describe (a combination of) boosts and rotations.

$$\text{Rotation: } \Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{R} \\ 0 & & & \\ 0 & & & \end{bmatrix}$$

↑
3x3 orthogonal matrix

$$\begin{aligned} \text{Boost: (in 1-direction)} \quad \Lambda &= \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c} (=v). \\ (\det \Lambda &= \gamma^2(1-\beta^2) = 1) \end{aligned}$$

Proper Lorentz-transf. can be continuously connected to the identity.

What is a scalar? A number $\in \mathbb{C}$ or a $\mathbb{C}$ -valued function which is invariant under Lorentz-transf. Eg. phase $p \cdot x = p_\mu x^\mu$ is invariant: $x^\mu \mapsto \Lambda^\mu_\alpha x^\alpha$, $p^\mu \mapsto \Lambda^\mu_\beta p^\beta$, $p_\mu x^\mu \mapsto g_{\mu\nu} \Lambda^\nu_\beta p^\beta \Lambda^\mu_\alpha x^\alpha$
 $= g_{\alpha\beta} x^\alpha p^\beta = x_\alpha x^\alpha$

What is a four-vector A^μ ? $A^\mu \mapsto \Lambda^\mu_\nu A^\nu$.

Normalization of $|\vec{p}\rangle$

Let $|\vec{p}\rangle$ be single particle states, $|\vec{p}\rangle \propto a_{\vec{p}}^\dagger |0\rangle$.

Consider a boost in the 3-direction of (E, p_1, p_2, p_3) .

Then $p_3' = \gamma(p_3 + \beta E)$

$E' = \gamma(E + \beta p_3)$

and

$$\delta(\vec{p} - \vec{q}) = \delta(p_1 - q_1) \delta(p_2 - q_2) \delta(p_3 - q_3)$$

$$\mapsto \delta(p_1' - q_1') \delta(p_2' - q_2') \delta(p_3' - q_3') \left| \frac{dp_3'}{dp_3} \right|$$

$$= \delta(\vec{p}' - \vec{q}') \gamma \left| 1 + \beta \frac{dE}{dp_3} \right|$$

$$= \delta(\vec{p}' - \vec{q}') \frac{\gamma}{E} |E + \beta p_3|$$

$$= \delta(\vec{p}' - \vec{q}') \frac{E'}{E}$$

so $E \delta(\vec{p} - \vec{q}) = E' \delta(\vec{p}' - \vec{q}')$.

$E \delta(\vec{p} - \vec{q})$ is Lorentz invariant.

Delta-func. property.

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

$$E^2 = \sum_i p_i^2 + m^2$$

$$2E dE = 2p_3 dp_3$$

How can we be sure about that, given that we have only considered boosts along 3-axis

Answer: May rotate coordinates

Choose normalization $|\vec{p}\rangle = \sqrt{2E_{\vec{p}}} a_{\vec{p}}^\dagger |0\rangle$

$$\Rightarrow \langle \vec{p} | \vec{q} \rangle = \sqrt{2E_{\vec{p}} 2E_{\vec{q}}} \langle 0 | a_{\vec{q}} a_{\vec{p}}^\dagger | 0 \rangle = 2E_{\vec{p}} (2\pi)^3 \delta(\vec{p} - \vec{q})$$

which is Lorentz-invariant.

And: $(1)_{1\text{-particle}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |\vec{p}\rangle \langle \vec{p}|$, $(1)_{\text{particle}} |\vec{q}\rangle = |\vec{q}\rangle$

Lorentz-invariant $\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} f(p)$

Integrals in the form $\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} f(p)$, where $f(p)$ is Lorentz-inv.
are proper Lorentz-inv.:

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} = \int \frac{d^4 p}{(2\pi)^4} 2\pi \frac{1}{2E_p} \delta(p^0 - E_p) = \int \frac{d^4 p}{(2\pi)^4} 2\pi \underbrace{\delta(p^2 - m^2)}_{\text{1D } \delta\text{-func.}} \Big|_{p^0 > 0}$$

Contribution from $\delta(p^2 - m^2)$ when $p^2 - \vec{p}^2 - m^2 = p^0^2 - E_p^2 = 0$
 $(p^0 - E_p)(p^0 + E_p) = 0$
 $2E_p (p^0 - E_p) = 0$ assuming $p^0 > 0$
 $\delta(ax) = \frac{1}{|a|} \delta(x)$

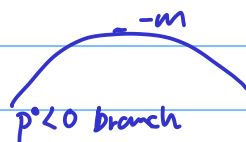
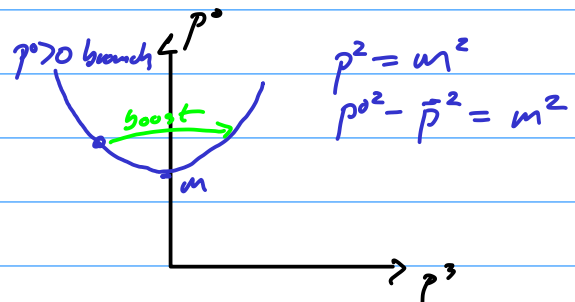
Lorentz-invariant because:

1) $d^4 p$ Lorentz-inv.: $d^4 p' = |J| d^4 p$, $J = \text{Jacobian determinant}$
 $J = \det \Lambda$ for transf. $x'^\mu = \Lambda^\mu_\nu x^\nu$

$$d^4 p' = d^4 p$$

Intuitively: Time dilation and length contraction cancel.

2) A proper Lorentz-transf. transforms $p^0 > 0$ into a p'^0 with $p'^0 > 0$, since it can be continuously connected to the identity.



$$\underline{D(x-y) = \langle 0 | \varphi(x) \varphi(y) | 0 \rangle}$$

PS: " $\varphi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-i\vec{p}\cdot\vec{x}} |\vec{p}\rangle$ is a particle at position $\vec{x}$." In nonrel. limit $E_p^2 = \vec{p}^2 + m^2 \approx m^2 = \text{const.}$, so then the interpretation is in line with QM, $|\vec{x}\rangle \propto \varphi(\vec{x})|0\rangle$. Furthermore $\underbrace{\langle 0 | \varphi(\vec{x}) | \vec{p} \rangle}_{(\langle \vec{x} | \text{ in QM})} = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_q} e^{i\vec{q}\cdot\vec{x}} \langle \vec{q} | \vec{p} \rangle = e^{i\vec{p}\cdot\vec{x}}.$

PS: $D(x-y)$ = "probability amplitude for a particle to propagate from x to y ". Why? Due to interpretation above.

$$\begin{aligned} D(x-y) &= \langle 0 | \varphi(x) \varphi(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2E_p 2E_q} e^{-ipx + iqy} \langle p | q \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \quad (\text{Lorentz-inv.}) \end{aligned}$$

We expect that $D(x-y) = 0$ for $x^0 - y^0 = 0$, since a particle cannot prop. from $\vec{x}$ to $\vec{y}$ instantaneously.

$$D(\vec{r}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}\cdot\vec{r}} = \frac{1}{2(2\pi)^2} \int_0^\infty dp \int_{-\cos\theta}^{\cos\theta} d\cos\theta \frac{p^2}{\sqrt{p^2+m^2}} e^{ipr\cos\theta}$$

$$(\vec{r} = \vec{x} - \vec{y})$$

$$= \frac{1}{2(2\pi)^2} \int_0^\infty dp \frac{p^2}{\sqrt{p^2+m^2}} \frac{e^{ipr} - e^{-ipr}}{ipr} = \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{p e^{ipr}}{\sqrt{p^2+m^2}}$$

$$\sim \frac{e^{-mr}}{\sqrt{r}} \text{ for } r \rightarrow \infty$$

$$\left(\int_{-\infty}^0 dp p = - \int_0^\infty dp p = - \int_0^\infty dp p \right)$$

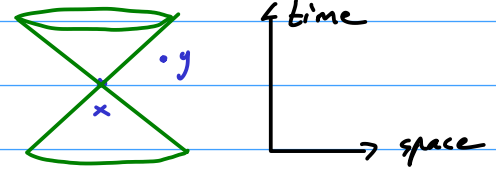
Wolfram Alpha Fourier transform, then asymptotic behavior of K_1 (Bessel func. second kind).

$\neq 0!$

Violation of causality? No. Particles have tails...

Causality

A measurement at x cannot affect a measurement at y , if $x - y$ is spacelike.



$$\begin{aligned}
 [\varphi(x), \varphi(y)] &= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2E_p 2E_q} [(a_p e^{-ipx} + a_p^\dagger e^{ipx}), (a_q e^{-iqy} + a_q^\dagger e^{iqy})] \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip(x-y)} - e^{ip(x-y)}) = D(x-y) - D(y-x) \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-i\vec{p} \cdot (\vec{x} - \vec{y})} - e^{i\vec{p} \cdot (\vec{x} - \vec{y})}) \\
 &\quad \text{[Lorentz transf. } x-y \mapsto \vec{x} - \vec{y} \text{ (possible since } x-y \text{ spacelike).} \\
 &\quad \downarrow \vec{p} \mapsto -\vec{p} \text{ last term} \\
 &= 0 \quad \text{for } x-y \text{ spacelike: Causality!}
 \end{aligned}$$

If $x = (t, 0, 0, z)$ is spacelike, then $|z/t| < 1$. With a boost $\beta = -z/t$, $x \mapsto x' = (t', 0, 0, z')$, where $t' = \gamma(t + \beta z) = 0$.

Klein-Gordon propagator

Let $x^0 > y^0$. $[\varphi(x), \varphi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip(x-y)} - e^{ip(x-y)})$

$D(x-y)$ $D(y-x)$
 $[\vec{p} \mapsto -\vec{p}]$

$$= \int \frac{d^3p}{(2\pi)^3} \left[\frac{e^{-ip(x-y)}}{2E_p} \Big|_{p^0=E_p} + \frac{e^{-ip(x-y)}}{-2E_p} \Big|_{p^0=-E_p} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{(-2\pi i)} \frac{1}{p^2 - m^2} e^{-ip(x-y)}$$

$p^0^2 - \vec{p}^2 - m^2 = p^0^2 - E_p^2$
 $= (p^0 - E_p)(p^0 + E_p)$

Retarded propagator

For any x^0, y^0 , define

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} = \begin{cases} [\varphi(x), \varphi(y)], & x^0 > y^0 \\ 0, & x^0 < y^0 \end{cases}$$

$$= \Theta(x^0 - y^0) [\varphi(x), \varphi(y)] = \Theta(x^0 - y^0) [D(x-y) - D(y-x)]$$

$\uparrow$ Heaviside func.

$iD_R(x-y)$ is a Green's function for the Klein-Gordon eq.:

$$i[\partial_\mu \partial^\mu + m^2] \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} = \int \frac{d^4p}{(2\pi)^4} \frac{(-1)}{p^2 - m^2} [(-ip_\mu)(-ip^\mu) + m^2] e^{-ip(x-y)}$$

$$= \delta(x-y).$$

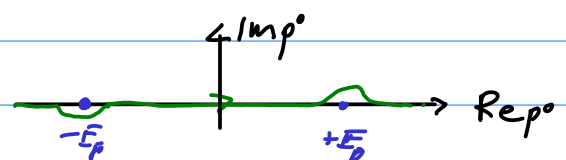
Retarded Green's function, since $D_R(x) = 0$ for $x^0 < 0$.

Feynman propagator

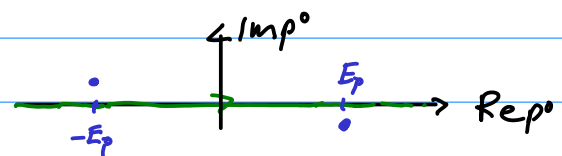
Define $D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)}$

Convenient form:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}$$



Feynman prescription



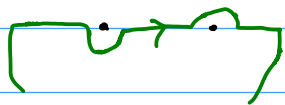
$$p^0^2 - E_p^2 + i\epsilon = (p^0 - \sqrt{E_p^2 - i\epsilon})(p^0 + \sqrt{E_p^2 - i\epsilon})$$

$$\sqrt{E_p^2 - i\epsilon} = E_p \sqrt{1 - \frac{i\epsilon}{E_p^2}} \approx E_p - \frac{i\epsilon}{2E_p}$$

Clearly we must have $i[\partial_\mu \partial^\mu + m^2] D_F(x-y) = \delta(x-y)$ (why?)

The Feynman propagator will be used a lot!

When $x^0 > y^0$:



$$D_F(x-y) = D(x-y)$$

When $x^0 < y^0$:



$$D_F(x-y) = D(y-x)$$

(+ sign because of orientation for contour.)

$$D_F(x-y) = \begin{cases} D(x-y), & x^0 > y^0 \\ D(y-x), & x^0 < y^0 \end{cases} = \langle 0 | T \overset{\leftarrow t}{\phi(x) \phi(y)} | 0 \rangle$$

↑ time ordering symbol, orders operators

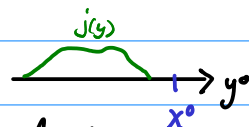
Interaction between a quantized KG field and a classical source

$$[\partial_\mu \partial^\mu + m^2] \phi(x) = j(x)$$

↑ classical field / scalar function, real

Let $\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ipx} + a_p^\dagger e^{ipx})$ be the free field.
Then

$$\phi(x) = \phi_0(x) + i \int d^4y D_R(x-y) j(y)$$



Let $x^0 > y^0$ for all y^0 with $j(y) \neq 0$ (wait until all of j is in the past): Then $D_R(x-y) = D(x-y) - D(y-x)$.

$$\begin{aligned} \phi(x) &= \phi_0(x) + i \int d^4y \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (e^{-ip(x-y)} - e^{ip(x-y)}) j(y) \\ &= \phi_0(x) + i \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [j(p) e^{-ipx} - j^*(p) e^{ipx}] \end{aligned}$$

$$\uparrow j(p) = \int d^4y e^{ip \cdot y} j(y)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[\left(a_p + \frac{i j(p)}{\sqrt{2E_p}} \right) e^{-ipx} + \left(a_p^\dagger - \frac{i j^*(p)}{\sqrt{2E_p}} \right) e^{ipx} \right]$$

a_p get "displaced" by $\frac{i}{\sqrt{2E_p}} j(p)$. Exercise: Show that $j(p)$ generates a coherent state.