

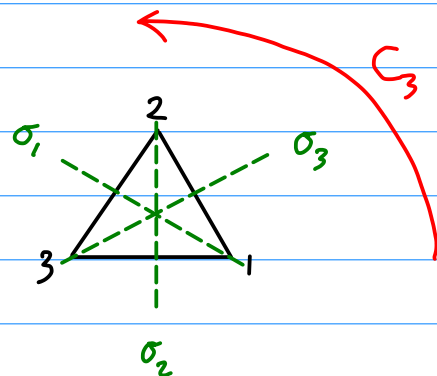
# Group theory

A group is a set  $G = \{g_i\}$  and an operation  $\circ$  ("group multiplication") s.t.:

- i) For any  $g_1, g_2 \in G$ ,  $g_1 \circ g_2 \in G$ .
- ii) For any  $g_1, g_2, g_3 \in G$ ,  $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$ .
- iii) There exists an identity element  $1 \in G$  s.t. for any  $g \in G$ ,  $1 \circ g = g \circ 1 = g$ .
- iv) Each element  $g \in G$  has an inverse  $g^{-1} \in G$  s.t.  $g^{-1} \circ g = g \circ g^{-1} = 1$

Example:  $C_{3v}$ , symmetry operations

$$C_{3v} = \{1, C_3, C_3^2, \sigma_1, \sigma_2, \sigma_3\}$$



Closed, since e.g.

$$\sigma_1 C_3 = \sigma_2$$

$$\sigma_1 \sigma_2 = C_3$$

$$\text{Inverse: } C_3^{-1} = C_3^2, \sigma_i^{-1} = \sigma_i$$

$$\text{Subgroup: } \{1, C_3, C_3^2\}$$

Example:  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ ,  $\circ = +$

- For  $g_1, g_2 \in \mathbb{Z}$ ,  $g_1 + g_2 \in \mathbb{Z}$
- Identity 0:  $0 + g = g + 0 = g$
- Inverse:  $-g$  is inverse to  $g$ .

Is  $\mathbb{R}$  a group? Yes, under addition, but not multiplication.

Group theory essential for studying number theory.  
Shor's algorithm...

Example: Minkowski space  $\mathbb{R}^4$  with  $o = +$  is a group, translation group.

$$x^\mu + y^\mu \in \mathbb{R}^4$$

$$x^\mu + (-x^\mu) \in \mathbb{R}^4$$

$$0 \in \mathbb{R}^4$$

Example: Lorentz group.  $\Delta_1, \Delta_2$  is a Lorentz transf. when  $\Delta_1, \Delta_2$  are.

## Lie groups

Group with elements depending on continuous parameters, s.t. group multiplication and inversion are smooth functions:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \dots \\ \dots & \dots \end{bmatrix}$$

Examples:

$GL(N)$ : All invertible  $N \times N$  matrices,  $o = \text{matrix mult.}$

$$\det(AB) = \det(A)\det(B) \Rightarrow \text{closed}$$

Subgroup:

$SU(N)$ : All unitary  $N \times N$  matrices with  $\det. 1$

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$$

Lorentz group

Poincare group (Lorentz transf. and translations)

## Group representations

"Represent group elements by matrices." matrices or operators

Def.: A map/function of group elements  $\rho(g)$  s.t.

if  $g_1, g_2 \in G$ , then

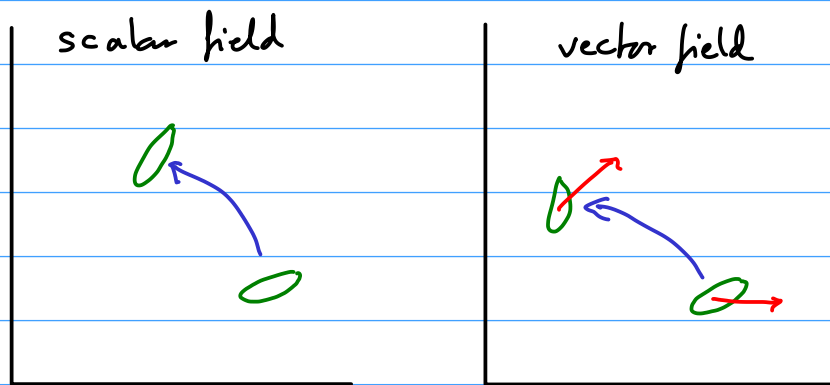
$$\rho(g_1 g_2) = \rho(g_1) \rho(g_2)$$

Ex:  $\rho(g) = 1$  for all  $g$ .

## Representations of the Lorentz group

- $U(\Lambda)$  : The unitary operators that act on quantum states as resulting from Lorentz-transf.  
Must have  $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)$
- $S(\Lambda)$  :  $4 \times 4$  matrix repr. acting on Dirac spinors.

# Transformations



$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu, \quad x \mapsto x' = \Lambda x$$

Scalar field  $\varphi(x)$  :

$$\begin{aligned} \varphi'(x') &= \varphi(x) \\ \varphi'(x') &= \varphi(\Lambda^{-1}x') & (\text{valid for all } x \\ \varphi'(x) &= \varphi(\Lambda^{-1}x) & \text{or all } x') \end{aligned}$$

Derivative of  $\varphi(x)$  :

$$\underbrace{\partial_\mu \varphi'(x)}_{\text{vector field}} = \partial_\mu \varphi(\Lambda^{-1}x) = \partial_\mu \varphi(\Lambda^{-1\nu} x^\nu)$$

Set  $u^\nu = \Lambda^{-1\nu} x^\mu$

Vector field  $A_\mu(x)$  :

$$\begin{aligned} A'_\mu(x) &= \Lambda^{-1\nu}_\mu A_\nu(\Lambda^{-1}x) \\ A'^\mu(x) &= \Lambda^{-1\nu}_\mu A^\nu(\Lambda^{-1}x) \\ &= g_{\nu\alpha} \Lambda^{-1\alpha}_\mu g^{\mu\beta} A^\nu(\Lambda^{-1}x) \\ &= \Lambda^\alpha_\mu A'^\nu(\Lambda^{-1}x) \end{aligned}$$

$$\left[ \begin{array}{l} g = \Lambda^T g' \Lambda \\ \Lambda^{-T} = g \Lambda g' \\ \Lambda^T = g' \Lambda^{-1} g \end{array} \right]$$

$$A' = \Lambda A(\Lambda^{-1}x)$$

For example, rotations:  $\vec{A}'(x) = \Lambda \vec{A}(\Lambda^{-1}x)$

# Lorentz group (Tong ch. 4-4.1)

To examine the structure of the Lorentz group,

we consider infinitesimal Lorentz-transf.: *This part is in problem set 4*

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu, \quad \Lambda = 1 + \omega, \quad \omega \text{ infinitesimal}$$

$$\Lambda^T g \Lambda = g \Rightarrow (1 + \omega^T) g (1 + \omega) = g$$

$$\Rightarrow g + g\omega + \omega^T g = g \Rightarrow g\omega = -\omega^T g \Rightarrow (g\omega)_{\mu\nu} = -(\omega^T g)_{\mu\nu}$$

$$\Rightarrow g_{\mu\sigma} \omega^\sigma_\nu = -\omega^\sigma_\mu g_{\sigma\nu} \Rightarrow \omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$\Rightarrow g^{\mu\alpha} g^{\nu\beta} \omega_{\alpha\beta} = -g^{\mu\alpha} g^{\nu\beta} \omega_{\beta\alpha} \Rightarrow \omega^{\mu\nu} = -\omega^{\nu\mu} \quad (\text{but } \omega^\mu_\nu = -\omega_{\nu}^\mu)$$

An antisymmetric 4x4 matrix has 6 indep. components:

$$\text{Basis: } (M^{\rho\sigma})^{\mu\nu} = g^{\rho\mu} g^{\sigma\nu} - g^{\sigma\mu} g^{\rho\nu}$$

*indices corresponding to those in  $\omega^{\mu\nu}$*   
*index the basis element, antisym. in  $\rho\sigma$*   
 $\rho, \sigma = 0, 1, 2, 3$

$$\begin{bmatrix} 0 & \cdot & \cdot & \cdot \\ & 0 & \cdot & \cdot \\ & & 0 & \cdot \\ & & & 0 \end{bmatrix}$$

$M^{\rho\sigma}$  is a basis, since for any  $\Omega_{\rho\sigma}$ :

$$\begin{aligned} \Omega_{\rho\sigma} (M^{\rho\sigma})^{\mu\nu} &= \Omega_{\rho\sigma} g^{\rho\mu} g^{\sigma\nu} - \Omega_{\rho\sigma} g^{\sigma\mu} g^{\rho\nu} \\ &= \Omega^{\mu\nu} - \Omega^{\nu\mu} = 2\Omega^{\mu\nu} \end{aligned}$$

Thus, if  $\Omega^{\mu\nu}$  is an arb. antisym. tensor, *antisymmetric*  
 $\frac{1}{2} \Omega_{\rho\sigma} (M^{\rho\sigma})^{\mu\nu} = \Omega^{\mu\nu}$

We have  $(M^{\rho\sigma})^\mu_\nu = g^{\rho\mu} \delta^\sigma_\nu - g^{\sigma\mu} \delta^\rho_\nu$  (in  $\Lambda^\mu_\nu$  we have lower index  $\nu$ .)

$$(M^{01})^\mu_\nu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline & & & \end{bmatrix}, \quad (M^{12})^\mu_\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

*(why is not  $M^{01}$  anti-symmetric?)*

Lorentz-algebra:

$$[M^{\rho\sigma}, M^{\tau\nu}] = g^{\sigma\tau} M^{\rho\nu} - g^{\rho\tau} M^{\sigma\nu} + g^{\rho\nu} M^{\sigma\tau} - g^{\sigma\nu} M^{\rho\tau}$$

An arbitrary infinitesimal Lorentz-transf.:

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \frac{1}{2} \Omega_{\rho\sigma} (M^{\rho\sigma})^\mu_\nu, \quad \Lambda = 1 + \frac{1}{2} \Omega_{\rho\sigma} M^{\rho\sigma}$$

$\Omega_{\rho\sigma}$  infinitesimal

An arb. (finite) Lorentz-transf. Let  $\Omega_{\rho\sigma}$  be finite (not infinitesimal)

$$\Lambda = \left(1 + \frac{\Omega_{\rho\sigma} \gamma^{\rho\sigma}}{n}\right)^n \xrightarrow{n \rightarrow \infty} e^{\frac{1}{2} \Omega_{\rho\sigma} \gamma^{\rho\sigma}}$$

$$\Lambda = e^{\frac{1}{2} \Omega_{\rho\sigma} \gamma^{\rho\sigma}}$$

The structure of the Lorentz group is determined by the Lorentz algebra through the Baker-Campbell-Hausdorff formula:

$$e^X e^Y = e^Z, \quad Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots$$

## Spinor representation

Clifford algebra: 4 matrices  $\gamma^\mu$ ,  $\mu = 0, 1, 2, 3$  with

$$\begin{cases} \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot 1 \\ \{A, B\} = AB + BA, \text{ anticommutator} \end{cases}$$

$$\gamma^0{}^2 = 1, \quad \gamma^i{}^2 = -1, \quad \gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \text{ for } \mu \neq \nu$$

A 4x4  
(chiral repr.)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

(each element is a 2x2 matrix)

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned} \{\sigma^i, \sigma^j\} &= 2\delta^{ij}, \quad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk} \sigma^k \\ \sigma^i \sigma^j &= \frac{[\sigma^i, \sigma^j] + \{\sigma^i, \sigma^j\}}{2} = \delta^{ij} + i\epsilon^{ijk} \sigma^k \end{aligned}$$

$$\text{Define } S^{\rho\sigma} = \frac{1}{4} [\gamma^\rho, \gamma^\sigma] = \begin{cases} 0, & \rho = \sigma \\ \frac{1}{2} \gamma^\rho \gamma^\sigma, & \rho \neq \sigma \end{cases} = \frac{1}{2} \gamma^\rho \gamma^\sigma - \frac{1}{2} g^{\rho\sigma} \quad (*)$$

$$[S^{\mu\nu}, \gamma^\rho] = \frac{1}{2} [\gamma^\mu \gamma^\nu, \gamma^\rho] = \frac{1}{2} \gamma^\mu \gamma^\nu \gamma^\rho - \frac{1}{2} \gamma^\rho \gamma^\mu \gamma^\nu$$

$$\begin{aligned} &\stackrel{\text{assume } \mu \neq \nu}{=} \frac{1}{2} \gamma^\mu \{\gamma^\nu, \gamma^\rho\} - \frac{1}{2} \gamma^\mu \gamma^\rho \gamma^\nu - \frac{1}{2} \{\gamma^\rho, \gamma^\mu\} \gamma^\nu + \frac{1}{2} \gamma^\rho \gamma^\mu \gamma^\nu \\ &= \gamma^\mu g^{\nu\rho} - g^{\rho\mu} \gamma^\nu. \quad \text{Works also for } \mu = \nu, \text{ since then } S^{\mu\mu} = 0. \end{aligned}$$

The matrices  $S^{\mu\nu}$  form a representation of the Lorentz-algebra, i.e., they obey the same algebra as  $M^{\mu\nu}$ .

$$[S^{\mu\nu}, S^{\rho\sigma}] = \frac{1}{2} [S^{\mu\nu}, \gamma^\rho \gamma^\sigma]$$

↑ assume  $\rho \neq \sigma$

$$= \frac{1}{2} [S^{\mu\nu}, \gamma^\rho] \gamma^\sigma + \frac{1}{2} \gamma^\rho [S^{\mu\nu}, \gamma^\sigma]$$

$$\uparrow [a, bc] = [a, b]c + b[a, c]$$

$$= \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \gamma^\sigma + \frac{1}{2} \gamma^\rho (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$$

$$= \frac{1}{2} \gamma^\rho (2S^{\mu\sigma} + g^{\mu\sigma}) - \frac{1}{2} \gamma^\sigma (2S^{\mu\rho} + g^{\mu\rho})$$

$$\frac{1}{2} \gamma^{\nu\sigma} (2S^{\rho\mu} + g^{\rho\mu}) - \frac{1}{2} \gamma^{\sigma\mu} (2S^{\rho\nu} + g^{\rho\nu})$$

$$\uparrow \gamma^\mu \gamma^\nu = 2S^{\mu\nu} + g^{\mu\nu} \text{ from (*)}$$

$$([S^{\rho\sigma}, S^{\tau\nu}] = \underbrace{-\gamma^{\nu\rho} S^{\sigma\tau}}_{+\gamma^{\nu\rho} S^{\sigma\tau}} + \underbrace{\gamma^{\sigma\tau} S^{\rho\nu}}_{-\gamma^{\sigma\tau} S^{\rho\nu}} - \gamma^{\nu\sigma} S^{\rho\tau} + \gamma^{\sigma\tau} S^{\rho\nu})$$

Let  $S[\Lambda] = e^{\frac{1}{2} \Omega_{\rho\sigma} S^{\rho\sigma}}$  for  $\Lambda = e^{\frac{1}{2} \Omega_{\rho\sigma} M^{\rho\sigma}}$   $[= \Lambda_{\frac{1}{2}} \text{ in } PS]$   $[S^{\rho\sigma}, S^{\tau\nu}] = g^{\sigma\tau} S^{\rho\nu} - g^{\sigma\rho} S^{\tau\nu} + g^{\rho\nu} S^{\sigma\tau} - g^{\sigma\nu} S^{\rho\tau}$   
 $[M^{\rho\sigma}, M^{\tau\nu}] = g^{\sigma\tau} M^{\rho\nu} - g^{\sigma\rho} M^{\tau\nu} + g^{\rho\nu} M^{\sigma\tau} - g^{\sigma\nu} M^{\rho\tau}$

$$\Lambda_3 = \Lambda_1 \Lambda_2 = e^{\frac{1}{2} \Omega'_{\rho\sigma} M^{\rho\sigma}} \cdot e^{\frac{1}{2} \Omega''_{\rho\sigma} M^{\rho\sigma}} = e^{\frac{1}{2} \Omega^3_{\rho\sigma} M^{\rho\sigma}}$$

$$S[\Lambda_1] S[\Lambda_2] = e^{\frac{1}{2} \Omega'^2_{\rho\sigma} S^{\rho\sigma}} \cdot e^{\frac{1}{2} \Omega''^2_{\rho\sigma} S^{\rho\sigma}} \stackrel{BCH}{=} e^{\frac{1}{2} \Omega^3_{\rho\sigma} S^{\rho\sigma}} = S[\Lambda_1 \Lambda_2]$$

The  $S[\Lambda]$  matrices form a representation of the Lorentz group.

$M^{\rho\sigma}$  and  $S^{\rho\sigma}$  are called generators of the Lorentz group and spinor repr., respectively.

Dirac spinor:

An object of 4 complex numbers  $\psi^\alpha(x)$  which transforms as

$$\psi^\alpha(x) \xrightarrow{\Lambda} S[\Lambda]^\alpha_\beta \psi^\beta(\Lambda^{-1}x) \quad (\text{while a four-v. transforms: } A^\mu(x) \mapsto \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x))$$

Is this a vector? No, since  $S[\Lambda] \neq \Lambda$ , as demonstrated by special cases:

# 1) Rotations:

Let  $\Omega_{0\nu} = \Omega_{\mu 0} = 0$  for all  $\mu, \nu$ .

$$S^{ij} = \frac{1}{2} \gamma^i \gamma^j = \frac{1}{2} \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix} = -\frac{i}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad (\text{for } i \neq j)$$

$$\epsilon \sigma^i \sigma^i = i \epsilon^{ijk} \sigma^k$$

Define  $\varphi^k$  by  $\Omega_{ij} = \epsilon_{ijk} \varphi^k$ . (E.g.  $\Omega_{12} = \varphi^3$ )

$$\frac{1}{2} \Omega_{ij} S^{ij} = \frac{1}{2} \epsilon_{ijk} \varphi^k \left( \frac{-i}{2} \right) \epsilon^{ijl} \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix} = \frac{-i}{4} \varphi^k 2 \sigma^k \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix}$$

$$= \frac{-i}{2} \varphi^k \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = -\frac{i}{2} \vec{\varphi} \cdot \vec{\sigma} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S[\Lambda] = \begin{pmatrix} e^{-\frac{i}{2} \vec{\varphi} \cdot \vec{\sigma}} & 0 \\ 0 & e^{\frac{i}{2} \vec{\varphi} \cdot \vec{\sigma}} \end{pmatrix}$$

For  $\vec{\varphi} = (0, 0, 2\pi)$ ,  $S[\Lambda] = \begin{pmatrix} e^{-i\pi\sigma^3} & 0 \\ 0 & e^{i\pi\sigma^3} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

On the other hand,

$$\Lambda = e^{\frac{1}{2} \Omega_{\mu\nu} M^{\mu\nu}} = e^{\frac{1}{2} \Omega_{12} M^{12} + \frac{1}{2} \Omega_{21} M^{21}}$$

$$= e^{\Omega_{12} M^{12}} = \exp \left[ \varphi^3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] = \exp \left[ -i \varphi^3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\varphi^3\sigma^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^{-i\varphi^3\sigma^2} = \cos(\varphi^3\sigma^2) - i \sin(\varphi^3\sigma^2)$$

$$= \cos(\varphi^3) - i \sigma^2 \sin(\varphi^3) = \begin{pmatrix} \cos \varphi^3 & -\sin \varphi^3 \\ \sin \varphi^3 & \cos \varphi^3 \end{pmatrix}$$

$$\epsilon(\sigma^2)^2 = 1, \sigma^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \text{rotation, angle } \varphi^3.$$

$$\left( \begin{aligned} (\Omega_{ij} M^{ij})^m_n &= -\epsilon_{ijk} \varphi^k (g^{im} \delta^j_n - g^{jm} \delta^i_n) \\ &= -\epsilon_{ink} \varphi^k g^{im} + \epsilon_{nj k} \varphi^k g^{jm} \\ (\Omega_{ij} M^{ij})_{mn} &= -\epsilon_{mnk} \varphi^k + \epsilon_{nmk} \varphi^k = 2\Omega_{mn} \\ (\Omega_{ij} M^{ij})^m_n &= 2\Omega^m_n \\ \Lambda &= e^{\frac{1}{2} \Omega_{ij} M^{ij}} = e^\Omega \end{aligned} \right)$$

In Tong the sign is incorrect;  $\varphi^i$  should be defined with opposite sign.

So for  $\varphi = 2\pi$  :  $\Lambda = 1$

$$S[\Lambda] = -1$$

$S[\Lambda]$  and  $\Lambda$  are different; the spinors are not vectors!  
Is it a problem that  $S[\Lambda] = -1$  even though  $\Lambda = 1$  (meaning no Lorentz transf.)? No, not if  $\psi$  is not an observable.

2) Boosts:  $\Omega_{ij} = 0$ ,  $\Omega_{i0} = -\Omega_{0i} = -\eta_i$   $\downarrow$  real (rapidity) ( $-\eta_i$  in Tong)

$$S^{0i} = \frac{1}{2} \gamma^0 \gamma^i = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$S[\Lambda] = e^{\frac{1}{2} \Omega_{0i} S^{0i} + \frac{1}{2} \Omega_{i0} S^{i0}} = e^{\Omega_{0i} S^{0i}} = \begin{pmatrix} e^{\vec{\eta} \cdot \vec{\sigma}/2} & 0 \\ 0 & e^{+\vec{\eta} \cdot \vec{\sigma}/2} \end{pmatrix}$$

For a boost along 1-direction,  $\Omega_{10} = -\Omega_{01} = -\eta$ , all other  $\Omega_{\alpha\beta} = 0$ .

$$\Lambda = e^{\frac{1}{2} \Omega_{\alpha\beta} M^{\alpha\beta}} = e^{\Omega_{01} M^{01}} = e^{\eta M^{01}} = \exp \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}$$

considering only the 0 and 1 rows and columns

$$= \exp(\eta \sigma^1) = \cosh \eta + \sigma^1 \sinh \eta$$

$$= \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} = \begin{pmatrix} \gamma & \gamma \beta \\ \gamma \beta & \gamma \end{pmatrix}$$

$$\text{so } \tanh \eta = \frac{\sinh \eta}{\cosh \eta} = \frac{\gamma \beta}{\gamma} = \beta$$

Rapidity  $\eta$  is additive:  $\Lambda_1 \Lambda_2 = \begin{pmatrix} \cosh \eta_1 & \sinh \eta_1 \\ \sinh \eta_1 & \cosh \eta_1 \end{pmatrix} \begin{pmatrix} \cosh \eta_2 & \sinh \eta_2 \\ \sinh \eta_2 & \cosh \eta_2 \end{pmatrix} = \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix}$   
 where  $\eta = \eta_1 + \eta_2$

For rotations  $S[\Lambda]$  is unitary. ( $\vec{\phi}$  real,  $\vec{\sigma}$  hermitian)

For boosts  $S[\Lambda]$  is not unitary.

So  $S[\Lambda]$  is not a unitary representation of the Lorentz group.

## Scalars, vectors, tensors in Dirac theory

We have a field  $\psi^\alpha(x)$  and a transf. law:

$$\psi^\alpha(x) \mapsto S[\Lambda]^\alpha_\beta \psi^\beta(\Lambda^{-1}x); \quad \psi \mapsto S[\Lambda]\psi$$

- $\psi^\alpha$  is not a vector
- $\psi^\dagger \psi$  is not a scalar:  
 $\psi^\dagger \psi \mapsto \psi^\dagger S[\Lambda]^\dagger S[\Lambda] \psi$ , and  $S[\Lambda]$  not unitary in general.

- Define  $\bar{\psi} = \psi^\dagger \gamma^0$ .

$$\bar{\psi} \mapsto \psi^\dagger S[\Lambda]^\dagger \gamma^0.$$

$$S^\dagger[\Lambda] = e^{\frac{1}{2} \Omega_{\mu\nu} (S^{\mu\nu})^\dagger}$$

$$(S^{\mu\nu})^\dagger = \frac{1}{4} [\gamma^{\sigma\mu} \gamma^{\nu\dagger} - \gamma^{\nu\mu} \gamma^{\sigma\dagger}]$$

For our repr. of  $\gamma$ -matrices:  $\gamma^{0\dagger} = \gamma^0$  and  $\gamma^{i\dagger} = -\gamma^i$ ,

$$\text{so } \gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu\dagger}$$

$$\begin{aligned} (S^{\mu\nu})^\dagger &= \frac{1}{4} \gamma^0 \gamma^\mu \gamma^0 \gamma^\nu \gamma^0 - \gamma^0 \gamma^\nu \gamma^0 \gamma^\mu \gamma^0 \\ &= \frac{1}{4} \gamma^0 [\gamma^\mu, \gamma^\nu] \gamma^0 = -\gamma^0 S^{\mu\nu} \gamma^0 = -S^{\mu\nu} \gamma^0 \end{aligned}$$

$$\underline{S[\Lambda]^\dagger} = e^{\frac{1}{2} \Omega_{\mu\nu} (S^{\mu\nu})^\dagger} = e^{-\frac{1}{2} \Omega_{\mu\nu} \gamma^0 S^{\mu\nu} \gamma^0} = \underline{\gamma^0 S[\Lambda]^{-1} \gamma^0}$$

$$\bar{\psi} \mapsto \psi^\dagger S[\Lambda]^\dagger \gamma^0 = \psi^\dagger \gamma^0 S[\Lambda]^{-1} \gamma^0 \gamma^0 = \bar{\psi} S[\Lambda]^{-1}$$

- $\bar{\psi} \psi$  is a scalar:

$$\bar{\psi} \psi \mapsto \bar{\psi} S[\Lambda]^{-1} S[\Lambda] \psi = \bar{\psi} \psi$$

- Is  $\gamma^\mu$  a vector?
- $\bar{\psi} \gamma^\mu \psi$  is a four-vector:  $\bar{\psi} \gamma^\mu \psi \mapsto \bar{\psi} S[\Lambda]^{-1} \gamma^\mu S[\Lambda] \psi$

For  $\Omega_{\rho\sigma}$  infinitesimal:  $\Lambda = 1 + \frac{1}{2} \Omega_{\rho\sigma} M^{\rho\sigma}$   
 $S[\Lambda] = 1 + \frac{1}{2} \Omega_{\rho\sigma} S^{\rho\sigma}$

Want  $(1 - \frac{1}{2} \Omega_{\rho\sigma} S^{\rho\sigma}) \gamma^\mu (1 + \frac{1}{2} \Omega_{\rho\sigma} S^{\rho\sigma}) = (1 + \frac{1}{2} \Omega_{\rho\sigma} M^{\rho\sigma})^\mu{}_\nu \gamma^\nu$   
 $- [S^{\rho\sigma}, \gamma^\mu] = (M^{\rho\sigma})^\mu{}_\nu \gamma^\nu$   
 $= (g^{\mu\rho} \delta^\sigma_\nu - g^{\sigma\mu} \delta^\rho_\nu) \gamma^\nu = g^{\mu\rho} \gamma^\sigma - g^{\sigma\mu} \gamma^\rho$

which has been proven above.

Two consecutive, inf. transf.:

$$\begin{aligned} (1 - \frac{1}{2} \Omega_{\rho\sigma} S^{\rho\sigma}) (1 - \frac{1}{2} \Omega_{\rho\sigma} S^{\rho\sigma}) \gamma^\mu (1 + \frac{1}{2} \Omega_{\rho\sigma} S^{\rho\sigma}) (1 + \frac{1}{2} \Omega_{\rho\sigma} S^{\rho\sigma}) \\ = (1 + \frac{1}{2} \Omega_{\rho\sigma} M^{\rho\sigma})^\mu{}_\nu (1 - \frac{1}{2} \Omega_{\rho\sigma} S^{\rho\sigma}) \gamma^\nu (1 + \frac{1}{2} \Omega_{\rho\sigma} S^{\rho\sigma}) \\ = (1 + \frac{1}{2} \Omega_{\rho\sigma} M^{\rho\sigma})^\mu{}_\kappa (1 + \frac{1}{2} \Omega_{\rho\sigma} M^{\rho\sigma})^\kappa{}_\nu \gamma^\nu \quad \text{etc.}, \end{aligned}$$

leads to  $S[\Lambda]^{-1} \gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu \gamma^\nu$

- $\bar{\psi} \gamma^\mu \gamma^\nu \psi$  is a tensor. (similar proof.)
- $\bar{\psi} \gamma^\mu \partial_\mu \psi$  is a scalar: ( $\gamma^\mu \partial_\mu$  is not!)

$$\begin{aligned} \bar{\psi} \gamma^\mu \partial_\mu \psi &\mapsto \bar{\psi} S[\Lambda]^{-1} \gamma^\mu \Lambda^{-1\nu}{}_\mu \partial_\nu S[\Lambda] \psi \\ &= \bar{\psi} S[\Lambda]^{-1} \gamma^\mu S[\Lambda] \Lambda^{-1\nu}{}_\mu \partial_\nu \psi \\ &= \bar{\psi} \Lambda^\mu{}_\kappa \gamma^\kappa \Lambda^{-1\nu}{}_\mu \partial_\nu \psi \\ &= \bar{\psi} \delta^\nu{}_\kappa \gamma^\kappa \partial_\nu \psi = \bar{\psi} \gamma^\mu \partial_\mu \psi \end{aligned}$$

$$\begin{aligned} \partial_\mu \psi &\mapsto \Lambda^{-1\nu}{}_\mu \partial_\nu \psi \\ S[\Lambda]^{-1} \gamma^\mu S[\Lambda] &= \Lambda^\mu{}_\nu \gamma^\nu \end{aligned}$$

# Dirac field

$$\text{Dirac Lagrangian: } \mathcal{L} = \bar{\psi}(x) (i \gamma^\mu \partial_\mu - m) \psi(x)$$

↑ Dirac spinor

Note:  $\bar{\psi}\psi$  and  $\bar{\psi}\gamma^\mu\partial_\mu\psi$  are scalars, and so is  $\mathcal{L}$ !

$$\text{Writing out elements: } \mathcal{L} = \bar{\psi}_\alpha (i \gamma^\mu_{\alpha\beta} \partial_\mu - m \delta_{\alpha\beta}) \psi_\beta(x)$$

(remember spinor indices are not spacetime indices)

Euler-Lagrange (consider  $\psi$  and  $\bar{\psi}$  as independent):

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi}_\alpha)} = 0$$

$$i \gamma^\mu_{\alpha\beta} \partial_\mu \psi_\beta - m \delta_{\alpha\beta} \psi_\beta = 0 \quad (\alpha, \beta \text{ spinor indices, so not lower/upper})$$

$$\boxed{(i \gamma^\mu \partial_\mu - m) \psi = 0} \quad \text{Dirac eq.}$$

[Lorentz-inv:

$$\begin{aligned} (i \gamma^\mu \partial_\mu - m) \psi &= 0 \rightarrow (i \gamma^\mu \Lambda^{-1\nu}_\mu \partial_\nu - m) S[\Lambda] \psi \\ &= S[\Lambda] S[\Lambda]^{-1} (i \gamma^\mu \Lambda^{-1\nu}_\mu \partial_\nu - m) S[\Lambda] \psi \\ &= S[\Lambda] (i \Lambda^\mu_\alpha \gamma^\alpha \Lambda^{-1\nu}_\mu \partial_\nu - m) \psi \\ &= S[\Lambda] (i \gamma^\nu \partial_\nu - m) \psi = 0 \end{aligned}$$

Define the slash:  $A_\mu \gamma^\mu \equiv \not{A}$

$$(i \not{\partial} - m) \psi = 0 \quad \text{Dirac eq.}$$

$$\not{A} \not{A} = A_\mu \gamma^\mu A_\nu \gamma^\nu = A_\mu A_\nu \frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$$

$$= g^{\mu\nu} A_\mu A_\nu = A_\mu A^\mu = A^2$$

$$\not{\partial} \not{\partial} = \partial_\mu \partial^\mu = \square$$

Each Dirac spinor component satisfies the Klein-Gordon eq.:

$$(i \not{\partial} + m) \underbrace{(i \not{\partial} - m)}_{=0} \psi = 0$$

$$(-\not{\partial} \not{\partial} - m^2) \psi = 0$$

$$(\square + m^2) \psi = 0, \quad \text{KG-eq.}$$

## Weyl spinors (PS p.44)

We had  $S[\Lambda] = \begin{pmatrix} e^{\frac{i}{2}\vec{\varphi}\cdot\vec{\sigma}} & 0 \\ 0 & e^{\frac{i}{2}\vec{\varphi}\cdot\vec{\sigma}} \end{pmatrix}$  for rotations,

and  $S[\Lambda] = \begin{pmatrix} e^{-\vec{\eta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{\vec{\eta}\cdot\vec{\sigma}/2} \end{pmatrix}$  for boosts.

Thus each  $1 \times 2$  block of  $\psi$  transforms separately:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad \begin{array}{l} \psi_L = \text{left-handed Weyl spinor} \\ \psi_R = \text{right-handed} \quad \text{--- " ---} \end{array}$$

So  $S[\Lambda]$  is a reducible representation of the Lorentz group.

$$\text{Dirac-eq.: } (i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

$$\left( \text{Recall: } \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \right)$$

The two components  $\psi_R$  and  $\psi_L$  are mixed by the  $m$ -term.

$$\text{For } m=0: \quad \begin{aligned} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L &= 0 \\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R &= 0 \end{aligned}$$

$$\text{Define } \sigma^\mu = (1, \vec{\sigma}), \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

$$\text{Then } \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\text{Dirac-eq.: } \begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

## Functions of matrices

How do we define/evaluate functions  $f(A)$ ,  $A$  square matrix?

1) Normal matrices ( $A^\dagger A = A A^\dagger$ ; satisfied for hermitian matrices, and unitary matrices):  $A = U \Lambda U^\dagger$  (diagonalization)  
 $\quad \quad \quad \uparrow$  diagonal matrix of eigenvalues

$$f(A) = U f(\Lambda) U^\dagger$$

With Dirac-notation:  $A = \sum_i \lambda_i |i\rangle\langle i|$  function of each eigenvalue, e.g.:  $f\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{pmatrix}$   
 $f(A) = \sum_i f(\lambda_i) |i\rangle\langle i|$

2) Power series: Assume  $f(x)$  has a power series

$$f(x) = \sum_n c_n x^n$$

Then

$$f(A) = \sum_n c_n A^n, \text{ provided it converges.}$$

If both 1) and 2) can be used ( $A$  normal;  $f$  analytic):

$$\begin{aligned} f(A) &= \sum_n c_n A^n = \sum_n c_n (U \Lambda U^\dagger)^n \\ &= U \sum_n c_n \Lambda^n U^\dagger = U f(\Lambda) U^\dagger = f_p(A) \end{aligned}$$

so they coincide.

Note: If normal  $A$  and  $B$  commute, then  $\sqrt{AB} = \sqrt{A} \sqrt{B}$ .

$$[A, B] = 0 \Rightarrow A = U \Lambda_A U^\dagger, B = U \Lambda_B U^\dagger \text{ (diagonalized in the same basis)}$$

$$\Rightarrow AB = U \Lambda_A \Lambda_B U^\dagger$$

$$\text{so } \sqrt{AB} = U \sqrt{\Lambda_A \Lambda_B} U^\dagger = U \sqrt{\Lambda_A} U^\dagger U \sqrt{\Lambda_B} U^\dagger = \sqrt{A} \sqrt{B}.$$

## Solutions to the Dirac-eq. (PS 3.3)

Let  $\psi(x) = u(p)e^{-ipx}$ .

Dirac eq.:  $(i\not{p} - m)\psi = 0$

$$(\not{p} - m)u = 0 \quad (*)$$

KG-eq.:  $(\square + m^2)\psi = 0$

$$(p^2 - m^2)u = 0, \quad p^2 = m^2$$

"on shell"

Consider the rest frame where  $p = (\pm m, 0)$ , and consider +.

Then  $(m\not{x} - m)u = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u = 0 \Rightarrow u = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \quad \xi^\dagger \xi = 1$$

$\xi$  is a 2c1 component spinor, describing the spin orientation of the Dirac sol.:  $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  spin up (along the 3-direction)  
 $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  spin down

Note that  $\psi$  (or  $u$ ) transforms under rotations

with  $S[\Lambda] = \begin{pmatrix} e^{-\frac{i}{2}\vec{p}\cdot\vec{\sigma}} & 0 \\ 0 & e^{\frac{i}{2}\vec{p}\cdot\vec{\sigma}} \end{pmatrix}$ , i.e.,  $\xi \mapsto e^{-\frac{i}{2}\vec{p}\cdot\vec{\sigma}} \xi$

as in QM.

Obtain Dirac solution in any frame by boosting:

$$u \mapsto u(p) = S[\Lambda]u = \sqrt{m} \begin{pmatrix} e^{-\vec{\eta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{\vec{\eta}\cdot\vec{\sigma}/2} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

Need a relation between  $\vec{\eta}$  and  $\vec{p}$ :

Let  $\vec{\eta} = (0, 0, \eta)$ ,  $\eta = \text{rapidity}$

$$u(p) = \sqrt{m} \begin{pmatrix} e^{-\eta\sigma^3/2} & 0 \\ 0 & e^{\eta\sigma^3/2} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = \sqrt{m} \begin{pmatrix} e^{-\eta\sigma^3/2} \xi \\ e^{\eta\sigma^3/2} \xi \end{pmatrix}$$

For a boost along 3-direction

$$\Lambda = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \quad \text{considering only the 0 and 3 rows and columns}$$

So  $\begin{pmatrix} E \\ p^3 \end{pmatrix} = \Lambda \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m \cosh \eta \\ m \sinh \eta \end{pmatrix}$ , giving the relation between  $\eta$  and  $\begin{pmatrix} E \\ p^3 \end{pmatrix}$ :

$$E + p^3 = m(\cosh \eta + \sinh \eta) = m e^\eta, \quad E - p^3 = m e^{-\eta}$$

Note that  $e^{-\eta \sigma^3} = \cosh \eta - \sigma^3 \sinh \eta = \frac{e^\eta + e^{-\eta}}{2} - \sigma^3 \frac{e^\eta - e^{-\eta}}{2}$

$$= e^\eta \frac{(1 - \sigma^3)}{2} + e^{-\eta} \frac{(1 + \sigma^3)}{2}$$

$$= \frac{1}{m} \begin{pmatrix} (E + p^3) \frac{1 - \sigma^3}{2} & (E - p^3) \frac{1 + \sigma^3}{2} \\ 0 & 0 \end{pmatrix} = \frac{1}{m} \begin{pmatrix} E - p^3 & 0 \\ 0 & E + p^3 \end{pmatrix}$$

$$= \frac{1}{m} p \cdot \sigma \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

since  $p \cdot \sigma = E \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - p^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} E - p^3 & 0 \\ 0 & E + p^3 \end{pmatrix}$ ; here  $E > p^3$ .

Conclusion:  $u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix}$ ,  $\sigma = (1, \vec{\sigma})$   
 $\bar{\sigma} = (1, -\vec{\sigma})$

Note that  $(p \cdot \sigma)(p \cdot \bar{\sigma}) = (E - \vec{p} \cdot \vec{\sigma})(E + \vec{p} \cdot \vec{\sigma}) = E^2 - (\vec{p} \cdot \vec{\sigma})^2$

$$= E^2 - p^i \sigma^i p^j \sigma^j = E^2 - p^i p^j \frac{\sigma^i \sigma^j + \sigma^j \sigma^i}{2}$$

$$= E^2 - p^i p^j \delta^{ij} = E^2 - \vec{p}^2 = p^2$$

$$= m^2$$

and  $m = \sqrt{(p \cdot \sigma)(p \cdot \bar{\sigma})} = \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}}$

! since  $[p \cdot \sigma, p \cdot \bar{\sigma}] = [E - \vec{p} \cdot \vec{\sigma}, E + \vec{p} \cdot \vec{\sigma}] = 0$

Verify that  $u(p)$  satisfies the Dirac eq.:

$$\begin{pmatrix} -m & i \sigma \cdot \partial \\ i \bar{\sigma} \cdot \partial & -m \end{pmatrix} \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix} e^{-i p x} = \begin{bmatrix} [-m \sqrt{p \cdot \sigma} + i \sigma \cdot (-i p) \sqrt{p \cdot \bar{\sigma}}] \xi \\ [i \bar{\sigma} \cdot (-i p) \sqrt{p \cdot \sigma} - m \sqrt{p \cdot \bar{\sigma}}] \xi \end{bmatrix} e^{-i p x}$$

$$= \begin{bmatrix} \sqrt{p \cdot \sigma} (-m + m) \xi \\ \sqrt{p \cdot \bar{\sigma}} (m - m) \xi \end{bmatrix} e^{-i p x} = 0$$

So  $\psi(x) = u(p) e^{-i p x}$ , with  $u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix}$  is a solution with four-momentum  $p$ .

Normalization:

Since  $\bar{\psi} = \psi^\dagger \gamma^0$ , we define  $\bar{u}(p) = u^\dagger(p) \gamma^0$ .

$$u^\dagger u = (\xi^\dagger \sqrt{p \cdot \sigma}, \xi^\dagger \sqrt{p \cdot \bar{\sigma}}) \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix} = \xi^\dagger (p \cdot \sigma + p \cdot \bar{\sigma}) \xi = 2E_p \xi^\dagger \xi$$

$$\bar{u} u = \dots \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\gamma^0} \dots = 2m \xi^\dagger \xi \quad (\text{Lorentz inv.})$$

For  $p = (E, 0, 0, p^3)$ ,  $p \cdot \sigma = \begin{pmatrix} E+p^3 & 0 \\ 0 & E-p^3 \end{pmatrix}$ ,  $p \cdot \bar{\sigma} = \begin{pmatrix} E+p^3 & 0 \\ 0 & E-p^3 \end{pmatrix}$

Let  $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (spin up):  $u_\uparrow(p) = \begin{bmatrix} \sqrt{E-p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E+p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix} \xrightarrow[p^3 \propto E]{\text{ultra-rel.}} \sqrt{2E} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Let  $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  (spin down):  $u_\downarrow(p) = \begin{bmatrix} \sqrt{E+p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E-p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} \rightarrow \sqrt{2E} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Define spin vector  $\vec{S} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$  (in QM:  $\vec{S} = \frac{\vec{\sigma}}{2}$ ), and

helicity  $h = \hat{p} \cdot \vec{S}$ ,

for  $\hat{p}$  along z-axis:  $h = S^3 = \frac{1}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$ ,  $u_\uparrow(p)$  and  $u_\downarrow(p)$  eigenvectors with eigenvalues  $+\frac{1}{2}$  and  $-\frac{1}{2}$ .

There are also solutions with  $\psi(x) = v(p) e^{+ipx}$  (neg. freq. solutions).

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix}, \quad s = 1, 2$$

$$\bar{u}^r(p) u^s(p) = 2m \delta^{rs}$$

$$u^{r\dagger}(p) u^s(p) = 2E_p \delta^{rs}$$

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{pmatrix}, \quad s = 1, 2$$

$$\bar{v}^r(p) v^s(p) = -2m \delta^{rs}$$

$$v^{r\dagger}(p) v^s(p) = 2E_p \delta^{rs}$$

The  $u$ 's and  $v$ 's are orthogonal in the following sense:

$$\bar{u}^r(p) v^s(p) = \bar{v}^r(p) u^s(p) = 0 \quad \text{and} \quad u^{r\dagger}(\vec{p}) v^s(-\vec{p}) = v^{r\dagger}(-\vec{p}) u^s(\vec{p}) = 0$$

$$\bar{u}^r(p) v^s(p) = (\xi^{r\dagger} \sqrt{p \cdot \sigma}, \xi^{r\dagger} \sqrt{p \cdot \bar{\sigma}}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{pmatrix} = \xi^{r\dagger} (-\sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} + \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma}) \eta^s = 0$$

$$u^{r\dagger}(\vec{p}) v^s(-\vec{p}) = (\xi^{r\dagger} \sqrt{p \cdot \sigma}, \xi^{r\dagger} \sqrt{p \cdot \bar{\sigma}}) \begin{pmatrix} \sqrt{p \cdot \bar{\sigma}} \eta^s \\ -\sqrt{p \cdot \sigma} \eta^s \end{pmatrix} = 0$$

## Spin sums

When calculating scattering cross section / detection rates, one must often sum over spins / polarizations.

$$\begin{aligned} \sum_{s=1,2} u^s(p) \bar{u}^s(p) &= \sum_s \begin{pmatrix} \sqrt{p \cdot \sigma} \zeta^s \\ \sqrt{p \cdot \sigma} \zeta^s \end{pmatrix} (\zeta^{s\dagger} \sqrt{p \cdot \sigma}, \zeta^{s\dagger} \sqrt{p \cdot \sigma}) \quad (\gamma^0 \text{ flips the two components}) \\ &= \begin{pmatrix} \sqrt{p \cdot \sigma} \\ \sqrt{p \cdot \sigma} \end{pmatrix} (\sqrt{p \cdot \sigma}, \sqrt{p \cdot \sigma}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \sqrt{p \cdot \sigma} & p \cdot \sigma \\ p \cdot \sigma & \sqrt{p \cdot \sigma} \sqrt{p \cdot \sigma} \end{pmatrix} \\ \underbrace{\sum_s \zeta^s \zeta^{s\dagger}}_{\substack{s=1,2 \\ (\sum_{s=1}^2 |s| > 1)}} = 1 &= \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \sigma & m \end{pmatrix} = m + p^0 \gamma^0 - p^i \gamma^i = \not{p} + m \\ \sum_s v^s(p) \bar{v}^s(p) &= \not{p} - m \quad (p \cdot \sigma = p_0 + p_i \sigma^i) \end{aligned}$$

$$\underline{\gamma^5}$$

$$\text{Define } \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma$$

$$\begin{aligned} \gamma^{5\dagger} &= -i \gamma^3 \gamma^2 \gamma^1 \gamma^0 = -i \cancel{\gamma^0} \gamma^3 \cancel{\gamma^0} \gamma^2 \cancel{\gamma^0} \gamma^1 \cancel{\gamma^0} \gamma^0 \\ &= -i \gamma^0 \gamma^3 \gamma^2 \gamma^1 \quad \underbrace{\gamma^{i\dagger} = \gamma^0 \gamma^i \gamma^0}_{\text{}} \\ &= +i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5 \Rightarrow \gamma^5 \text{ hermitian} \\ \underbrace{\{\gamma^\mu, \gamma^\nu\}}_{\text{}} &= 2g^{\mu\nu} \end{aligned}$$

$$\begin{aligned} (\gamma^5)^2 &= 1 \quad \text{since } (\gamma^5)^2 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \dots = 1 \\ \{\gamma^5, \gamma^\mu\} &= 0 \quad \text{since } \gamma^5 \gamma^\mu = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\mu = -\gamma^\mu \gamma^5 \end{aligned}$$

In the exercise you prove that the symmetry  $\psi(x) \mapsto e^{-i\alpha} \psi(x)$  leads to the conserved Noether current  $j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x)$ .

Similarly  $\psi(x) \mapsto e^{-i\alpha \gamma^5} \psi(x)$  leads to a conserved current  $j^\mu = \bar{\psi}(x) \gamma^\mu \gamma^5 \psi(x)$  when  $m=0$ : ↙ chiral transformation

$$\begin{aligned} \mathcal{L} &= \bar{\psi} i \gamma^\mu \partial_\mu \psi = i \psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \psi \\ &\rightarrow i \psi^\dagger e^{i\alpha \gamma^5} \gamma^0 \gamma^\mu \partial_\mu e^{-i\alpha \gamma^5} \psi = i \psi^\dagger \gamma^0 e^{-i\alpha \gamma^5} e^{i\alpha \gamma^5} \gamma^\mu \partial_\mu \psi \\ &= i \bar{\psi} \gamma^\mu \partial_\mu \psi \quad \text{[}\{\gamma^5, \gamma^\mu\} = 0\text{]} \end{aligned}$$

so  $\mathcal{L}$  is unchanged. Then, since  $\psi \mapsto e^{-i\alpha \gamma^5} \psi = (1 - i\alpha \gamma^5) \psi$  to first order, and  $\mathcal{L} = i \bar{\psi}_\alpha \gamma^\mu_{\alpha\beta} \partial_\mu \psi_\beta$ ,

$$\begin{aligned} j^\mu \gamma^5 &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \Delta \psi_\beta + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} \Delta \bar{\psi}_\alpha = i \bar{\psi}_\alpha \gamma^\mu_{\alpha\beta} (-i\alpha (\gamma^5 \psi)_\beta) \\ &= \alpha \bar{\psi} \gamma^\mu \gamma^5 \psi \end{aligned}$$

$$\partial_\mu j^\mu \gamma^5 = (\partial_\mu \bar{\psi}) \gamma^\mu \gamma^5 \psi + \bar{\psi} \gamma^\mu \gamma^5 \partial_\mu \psi$$

Dirac:  $i \gamma^\mu \partial_\mu \psi = 0$

$$\begin{aligned} 0 &= \partial_\mu \psi^\dagger \gamma^0 \gamma^\mu \gamma^5 = -i \partial_\mu \psi^\dagger \gamma^0 \gamma^\mu \gamma^0 - -i \partial_\mu \bar{\psi} \gamma^\mu \gamma^0 \\ 0 &= \partial_\mu \bar{\psi} \gamma^\mu \end{aligned}$$

$= 0$

Interpretation of  $j^\mu \gamma^5$ :  $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ , so  $\frac{1-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  projectors onto left-handed / right-handed Weyl spinors

$$\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Def:  $j_L^\mu = \bar{\psi} \gamma^\mu \frac{1-\gamma^5}{2} \psi$  depends only on  $\psi_L$ ; el. current densities of left-handed

$j_R^\mu = \bar{\psi} \gamma^\mu \frac{1+\gamma^5}{2} \psi$  — " —  $\psi_R$ ; — " — right-handed

$\hookrightarrow \psi + \gamma^0 \gamma^\mu, \gamma^0 \gamma^\mu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$

$j^\mu \gamma^5$  is a vector:  $j^\mu \gamma^5 = \bar{\psi} \gamma^\mu \gamma^5 \psi \mapsto \bar{\psi} S[\Lambda]^{-1} \gamma^\mu \gamma^5 S[\Lambda] \psi$

$= \Lambda^\mu_\nu \bar{\psi} \gamma^\nu \gamma^5 \psi$  since  $[\gamma^5, S[\Lambda]] = 0$ , as resulting from  $[\gamma^5, S^{\mu\nu}] = 0$ .

$\bar{\psi} \psi$ ,  $\bar{\psi} \gamma^\mu \psi$ ,  $\bar{\psi} \gamma^\mu \gamma^5 \psi$ ,  $\bar{\psi} \gamma^\mu \gamma^\nu \psi$ , in general  $\bar{\psi} T \psi$ , are called Dirac field bilinears: Observables in Dirac theory.

## Quantization of the Dirac field

From  $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$  we have

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger, \quad \pi_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0$$

and the Hamiltonian

$$\begin{aligned} H &= \int d^3x [\pi(\vec{x}) \dot{\psi}(\vec{x}) - \mathcal{L}] = \int d^3x [i\cancel{\psi^\dagger} \dot{\psi} - i\cancel{\psi^\dagger} \psi - i\bar{\psi} \gamma^i \partial_i \psi + m\bar{\psi}\psi] \\ &= \int d^3x \bar{\psi} (-i \vec{\gamma} \cdot \vec{\nabla} + m) \psi \end{aligned}$$

We write  $\psi(x)$  as a superposition of solutions to the D. eq.:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a_p^s u^s(p) e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ipx})$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (b_p^s \bar{v}^s(p) e^{-ipx} + a_p^{s\dagger} \bar{u}^s(p) e^{ipx})$$

Canonical quantization would mean

$$[\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})] = \delta(\vec{x} - \vec{y}) \delta_{\alpha\beta} \quad (\text{equal times})$$

Leads to ladder op. properties of  $a_p^s$  and  $b_p^s$ .

Similarly to the complex KG case, that would lead to

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s (E_p a_p^{s\dagger} a_p^s - E_p b_p^{s\dagger} b_p^s)$$

↑ the  $v$ -solutions leads to negative energy.

Also, we would obtain the possibility of creating several particles in the same mode,  $(a_p^{s\dagger})^n |0\rangle$ .

Since we want to describe fermions, we impose

$$\{\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})\} = \delta(\vec{x} - \vec{y}) \delta_{\alpha\beta} \quad (\text{equal times}) \quad (*)$$

$$\{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} = 0,$$

i.e. anticommutators rather than commutators.

This leads to

$$\{a_p^r, a_q^{s\dagger}\} = \{b_p^r, b_q^{s\dagger}\} = (2\pi)^3 \delta(\vec{p} - \vec{q}) \delta^{rs}, \quad (**)$$

$$\{a_p^r, a_q^s\} = \{b_p^r, b_q^s\} = \{a_p^r, b_q^s\} = 0$$

so for example  $(a_p^{s\dagger})^2 = 0$  (which demonstrate the fermionic property).

## Anti-commutators

- Linearity :  $\{\alpha A + \beta B, C\} = (\alpha A + \beta B)C + C(\alpha A + \beta B) = \alpha \{A, C\} + \beta \{B, C\}$   
(also in the second factor, of course).
- $\{A, B\} = \{B, A\}$ .
- $\{A, BC\} \neq \{A, B\}C + B\{A, C\}$  in general (although valid for  $[ ]$ ).  
But if  $\{A, C\} = 0$ , then  $\{A, BC\} = [A, B]C$  :  
$$\begin{aligned}\{A, BC\} &= ABC + BCA = ABC - BAC + BAC + BCA \\ &= [A, B]C + B\{A, C\} = \{BC, A\}\end{aligned}$$
- $[A, BC] = [A, B]C + B[A, C]$   
$$= \{A, B\}C - B\{A, C\} \quad (= ABC + BAC - BAC - BCA)$$
- $[BC, A] = B\{A, C\} - \{A, B\}C$

Show that (\*\*) follows from (\*\*):

$$\begin{aligned}
 \{\psi(\vec{x}), \psi^\dagger(\vec{y})\} &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left[ u_\alpha^s(p) u_\beta^{st}(p) e^{i\vec{p}\vec{y} - i\vec{p}\vec{x}} + v_\alpha^s(p) v_\beta^{st}(p) e^{-i\vec{p}\vec{y} + i\vec{p}\vec{x}} \right] \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ [\not{p} + m] e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} + [\not{p} - m] e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \right\} \delta^\alpha_\beta \\
 &\quad \left[ \sum_s u^s(p) \bar{u}^s(p) = \not{p} + m, \sum_s u^s(p) u^{st}(p) = \not{p} \delta^s + m \gamma^0 \right. \\
 &\quad \left. \sum_s v^s(p) v^{st}(p) = \not{p} \delta^s - m \gamma^0 \right] \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ E_p \delta^0 - \gamma^i p^i + m + E_p \delta^0 + \gamma^i p^i - m \right\} \delta^\alpha_\beta e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \\
 &= \delta(\vec{x}-\vec{y}) \delta_{\alpha\beta}
 \end{aligned}$$

(Note that for example,  $[u^s(p) u^{st}(p)]_{\alpha\beta} = u_\alpha^s(p) u_\beta^{st}(p)$ .

$$\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} \cdot & \dots & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \end{bmatrix}$$

Furthermore:

$$H = \int d^3x \bar{\psi} (-i \vec{\gamma} \cdot \vec{\nabla} + m) \psi = \dots = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p (a_p^{st} a_p^s + b_p^{st} b_p^s)$$

by using orthogonality of spinors and dropping an infinite constant.

$$\vec{P} = \int d^3x \psi^\dagger (-i \vec{\nabla}) \psi = \dots = \int \frac{d^3p}{(2\pi)^3} \sum_s \vec{p} (a_p^{st} a_p^s + b_p^{st} b_p^s)$$

( $P^i = - \int \pi \partial_i \varphi d^3x$ , and a sum when more than one field.)

$$\begin{aligned}
 Q &= \int d^3x \bar{\psi} \gamma^0 \psi = \int d^3x \psi^\dagger \psi \quad (j^\mu = \bar{\psi} \gamma^\mu \psi \text{ conserved current}) \\
 &= \int \frac{d^3p}{(2\pi)^3} \sum_s (a_p^{st} a_p^s - b_p^{st} b_p^s)
 \end{aligned}$$

$$\text{We have } [H, a_p^s] = \int \frac{d^3p}{(2\pi)^3} E_p [a_p^{st} a_p^s, a_p^s] = \int \frac{d^3p}{(2\pi)^3} E_p a_p^{st} \cdot 0 - \int \frac{d^3p}{(2\pi)^3} E_p (2\pi)^3 \delta(\vec{p}-\vec{q}) a_p^s$$

since  $\{a_p^r, b_q^{st}\} = 0$  and  $\{a_p^r, b_q^s\} = 0$

$$= -E_p a_p^s$$

$$[H, a_p^{st}] = +E_p a_p^{st}$$

Why ladder?  $(H a - a H) | \psi \rangle = -E_p a | \psi \rangle$

$$(H a - a E) | \psi \rangle = -E_p a | \psi \rangle$$

$$H \underline{a} | \psi \rangle = (E - E_p) a | \psi \rangle$$

From the form of H, eigenvalues must be positive.

Thus there is a state  $|\psi\rangle$  s.t.  $a|\psi\rangle = 0$ .

Thus both  $a_p^{st}$  and  $b_p^{st}$  create particles with energy  $+E_p$  and momentum  $\vec{p}$ , fermions anti-fermions

while the charge is opposite. But since  $(a_p^{st})^2 = (b_p^{st})^2 = 0$ , only one particle can be created in each mode  $(\vec{p}, s)$ .

Let  $|\vec{p}, s\rangle \equiv \sqrt{2E_p} a_p^{st} |0\rangle$  be called single-particle (fermion) states.

$$\langle \vec{p}, r | \vec{q}, s \rangle = 2E_p \langle 0 | a_p^r a_p^{st} | 0 \rangle = 2E_p 2\pi \delta(\vec{p} - \vec{q}) \delta^{rs} \quad (\text{Lorentz-inv.})$$

Similarly  $\sqrt{2E_p} b_p^{st} |0\rangle$  is a single anti-fermion.

## Angular momentum / spin

Classical Dirac field  $\psi(x) \mapsto \psi'(x) = S[\Lambda] \psi(\Lambda^{-1}x)$  under Lorentz-transf.  
 Consider a rotation about the 3-axis (to use Noether's theorem):

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{\text{infinitesimal}}{\approx} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S[\Lambda] = \begin{pmatrix} e^{-i\frac{\theta}{2}\sigma^3} & 0 \\ 0 & e^{-i\frac{\theta}{2}\sigma^3} \end{pmatrix} \downarrow \begin{pmatrix} 1 - i\frac{\theta}{2}\sigma^3 & 0 \\ 0 & 1 - i\frac{\theta}{2}\sigma^3 \end{pmatrix} = 1 - \frac{i}{2}\theta \Sigma^3, \quad \Sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

$$\begin{aligned} \delta\psi &= \psi'(x) - \psi(x) = \left(1 - \frac{i}{2}\theta \Sigma^3\right) \psi(t, x + \theta y, y - \theta x, z) - \psi(t, x, y, z) \\ &= \left(1 - \frac{i}{2}\theta \Sigma^3\right) [\psi + \theta y \partial_x \psi - \theta x \partial_y \psi] - \psi \\ &= [\theta y \partial_x - \theta x \partial_y - \frac{i}{2}\theta \Sigma^3] \psi = \theta \Delta\psi, \quad \Delta\psi = [y \partial_x - x \partial_y - \frac{i}{2}\Sigma^3] \psi \end{aligned}$$

Charge component of conserved Noether 4-current:

$$j^0 = \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} \Delta\psi \quad \begin{array}{l} \text{no term } j^0 \text{ since } \mathcal{L} \text{ scalar, so} \\ \mathcal{L}(x) \mapsto \mathcal{L}(\Lambda^{-1}x) = \mathcal{L}(x) + \theta \Delta\mathcal{L}(x), \\ \Delta\mathcal{L} = [y \partial_x - x \partial_y] \mathcal{L} = \partial_x(y\mathcal{L}) + \partial_y(-x\mathcal{L}) \\ = \partial_\mu J^\mu, \text{ with } J^\mu = (0, y\mathcal{L}, -x\mathcal{L}, 0) \end{array}$$

$$= -i\psi^\dagger (x \partial_y - y \partial_x + \frac{i}{2} \Sigma^3) \psi$$

$$\vec{x} \times (-i\vec{\nabla}) = -i \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ \partial_x & \partial_y & -i\partial_z \end{vmatrix}$$

= 0 for particle at rest ( $\vec{p}=0$ )  
 For non-rel. fermions  
 orbital angular momentum, spin

Noether's Theorem  $\Rightarrow \vec{J} = \int d^3x \psi^\dagger \left( \vec{x} \times (-i\vec{\nabla}) + \frac{1}{2} \vec{\Sigma} \right) \psi$   
 is conserved (classical Dirac field). Here  $\Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$ .

Spin:  $J_z = \frac{1}{2} \int d^3x \psi^\dagger \Sigma^3 \psi$

Want to find the eigenv. of  $J_z$  for a one-particle zero momentum state,  
 $a_0^{s\dagger} |0\rangle$ :  $J_z a_0^{s\dagger} |0\rangle = [J_z, a_0^{s\dagger}] |0\rangle$

$J_z |0\rangle = 0$ , because:  $[J_z, H] = 0$  (proved from Poincare symmetry),  
 so  $|0\rangle$  (nondegenerate) must be an eigenstate of  $J_z$ :

$J_z |0\rangle = j_z |0\rangle$ . But  $j_z = 0$  by symmetry;  $\langle 0 | J_z | 0 \rangle$  must be 0.

$$J_z = \frac{1}{2} \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_{p'}}} e^{-i\vec{p}' \cdot \vec{x} + i\vec{p} \cdot \vec{x}} \quad (\text{let } x^0 = 0)$$

$$\cdot \sum_{r, r'} \left[ \underline{a_{p'}^{r'\dagger} u^{r'\dagger}(\vec{p}')} + b_{-\vec{p}'}^{r'} v^{r'}(-\vec{p}') \right] \Sigma^3 \left[ \underline{a_{\vec{p}}^r u^r(\vec{p})} + b_{-\vec{p}}^{r\dagger} v^r(-\vec{p}) \right]$$

(  $\int d^3x$  leads to  $\delta(\vec{p}' - \vec{p})$  )  
 all other combinations give zero or annihilates  $|0\rangle$ .

$$[a_{\vec{p}}^{r'\dagger} a_{\vec{p}}^r, a_0^{s\dagger}] = a_{\vec{p}}^{r'\dagger} (2\pi)^3 \delta^{rs} \delta(\vec{p}-0)$$

Remember  $[BC, A] = B\{A, C\} - \{A, B\}C$

$$[J_z, a_0^{s\dagger}] |0\rangle = \frac{1}{4m} \sum_r \left( u^{r\dagger}(0) \Sigma^3 u^s(0) \right) a_0^{r\dagger} |0\rangle$$

$$= \frac{1}{4} \sum_r \begin{pmatrix} \xi^{r\dagger} & \xi^{r\dagger} \end{pmatrix} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} = \frac{1}{2} \sum_r \xi^{r\dagger} \sigma^3 \xi^s$$

$$= \pm \frac{1}{2} \quad \left[ + \text{ for } \xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; - \text{ for } \xi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

So  $J_z a_0^{s\dagger} |0\rangle = \pm \frac{1}{2} a_0^{s\dagger} |0\rangle$

Similarly

$$J_z b_0^{s\dagger} |0\rangle = \mp \frac{1}{2} b_0^{s\dagger} |0\rangle \quad [b_{-\vec{p}}^{r'} b_{-\vec{p}}^{r'\dagger}, b_0^{s\dagger}] = \mp b_{-\vec{p}}^{r'} \delta^{rs} (2\pi)^3 \delta(-\vec{p})$$

| Summary:          | state                                                  | momentum  | energy        | spin              | charge |
|-------------------|--------------------------------------------------------|-----------|---------------|-------------------|--------|
| Single fermion    | $\sqrt{2E_{\vec{p}}} a_{\vec{p}}^{s\dagger}  0\rangle$ | $\vec{p}$ | $E_{\vec{p}}$ | $\pm \frac{1}{2}$ | $+1$   |
| Single anti-ferm. | $\sqrt{2E_{\vec{p}}} b_{\vec{p}}^{s\dagger}  0\rangle$ | $\vec{p}$ | $E_{\vec{p}}$ | $\mp \frac{1}{2}$ | $-1$   |

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a_p^s u^s(p) e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ipx})$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (b_p^s \bar{v}^s(p) e^{-ipx} + a_p^{s\dagger} \bar{u}^s(p) e^{ipx})$$

## Dirac propagator

$\langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle$  Probability amplitude of creating a ferm. at  $y$  and detecting a fermion at  $x$ .

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_q}} \sum_s u_\alpha^s(p) e^{-ipx} \bar{u}_\beta^s(q) e^{iqy} \langle 0 | a_p^s a_q^{s\dagger} | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s u_\alpha^s(p) \bar{u}_\beta^s(p) e^{-ip(x-y)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (\not{p} + m)_{\alpha\beta} e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)_{\alpha\beta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$  for KG case

$$\langle 0 | \bar{\psi}_\beta(y) \psi_\alpha(x) | 0 \rangle = \dots = -(\not{\partial}_x + m)_{\alpha\beta} D(y-x)$$

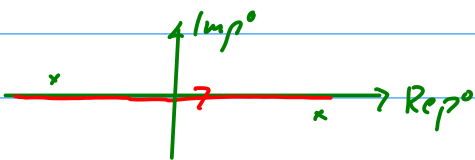
Opposite sign of  $m$  due to  $\bar{\psi} \psi = \bar{\psi} \psi$ . Opposite sign of  $\not{x}$  because of sign in  $e^{ipx - iqy}$  due to  $x \rightarrow y, y \rightarrow x$ .

Define time-ordering of spinor fields such that there is an extra — when operators are interchanged, i.e.:

$$T(\psi_A(x) \psi_B(y)) = \begin{cases} \psi_A(x) \psi_B(y), & x^0 > y^0 \\ -\psi_B(y) \psi_A(x), & x^0 < y^0 \end{cases}$$

Def Feynman prop.  $S_F(x-y) = \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle$

$$= \begin{cases} (\not{\partial}_x + m) D(x-y), & x^0 > y^0 \\ (\not{\partial}_x + m) D(y-x), & x^0 < y^0 \end{cases}$$



$$= (\not{\partial}_x + m) D_F(x-y)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

KG:  $D_F(x-y) = \langle 0 | T \phi(x) \phi(y) | 0 \rangle$

$$= \begin{cases} D(x-y), & x^0 > y^0 \\ D(y-x), & x^0 < y^0 \end{cases}, \quad D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}$$

# Discrete symmetries of Dirac theory

Any rel. theory should look the same in all frames.

Invariant under proper Lorentz transf. which can be continuously connected to the identity.

We also have discrete Lorentz transf.:

$$P: (t, \vec{x}) \mapsto (t, -\vec{x})$$

$$T: (t, \vec{x}) \mapsto (-t, \vec{x})$$

In addition we will consider charge conjugation

$C$ : interchange fermions and anti-fermions.

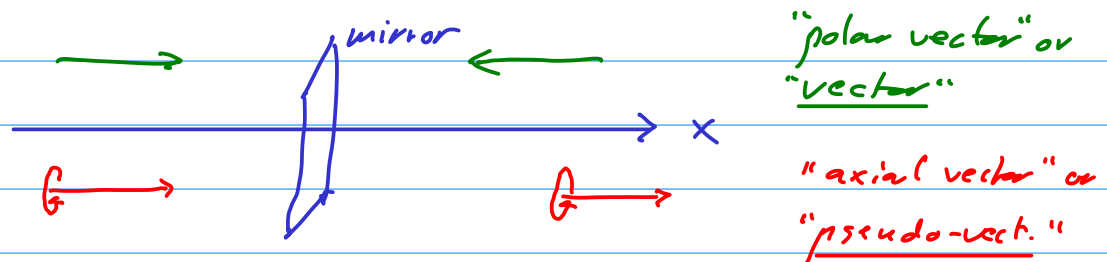
## Parity $P$

3D:

Can be implemented as a reflection about the  $yz$ -plane ( $x \mapsto -x$ ) followed by a  $\pi$  rotation about the  $x$ -axis:

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Rotation is a proper Lorentz-transf. so consider reflection:



Example: Position  $\vec{r}$ , velocity  $\vec{v}$ , forces  $\vec{F}$ ,  $\vec{E}$ , ...

Example: Torque  $\vec{T} = \vec{r} \times \vec{F}$ ,  $\vec{L} = \vec{r} \times \vec{p}$ ,  $\vec{B}$  (since  $\vec{F} = q \vec{v} \times \vec{B}$ ), ...  
(or  $\vec{v} \times \vec{E} = -\partial_t \vec{B}$ )

On the quantum state  $P$  should be implemented as a unitary operator  $U(P)$  (but just called  $P$ ).

Expect:

$$P a_p^{st} |0\rangle = \eta_a^* a_{-\vec{p}}^{st} |0\rangle \quad (\text{flip } \vec{p} \text{ but not } s \text{ of a particle})$$

± phase factor

$$P a_p^{st} P = \eta_a^* a_{-\vec{p}}^{st}, \quad P b_p^{st} P = \eta_b^* b_{-\vec{p}}^{st}$$

$$P a_p^s P = \eta_a a_{-\vec{p}}^s, \quad P b_p^s P = \eta_b b_{-\vec{p}}^s$$

$P^2: P^2 a_p^s P^2 = \eta_a^2 a_{\vec{p}}^s$  Strictly, this is not necessary; only nec. that  $\langle \text{observables} \rangle$  invariant.

Dirac field bilinears  
↓

We require  $\eta_a^2 = 1 = \eta_b^2$ , which is consistent with  $P^2 = 1$  and  $PP^\dagger = P^\dagger P = 1$ , and therefore  $P^\dagger = P$ .

$P$  transforms quantum states as  $|\Phi\rangle \mapsto P|\Phi\rangle$ , so it transforms operators as  $\psi(x) \mapsto P\psi(x)P$ :

$$P\psi(x)P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( \eta_a a_{-\vec{p}}^s u^s(p) e^{-ipx} + \eta_b^* b_{-\vec{p}}^{st} v^s(p) e^{ipx} \right)$$

Change variable to  $\tilde{p} = (p^0, -\vec{p})$ . Define  $\tilde{x} = (x^0, -\vec{x})$

Note  $p \cdot x = \tilde{p} \cdot \tilde{x}$

$$\tilde{p} \cdot \sigma = p \cdot \bar{\sigma} \quad \text{and} \quad \tilde{p} \cdot \bar{\sigma} = p \cdot \sigma \quad [\sigma = (1, \vec{\sigma}), \bar{\sigma} = (1, -\vec{\sigma})]$$

$$\text{So } u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \\ \sqrt{p \cdot \bar{\sigma}} \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p} \cdot \bar{\sigma}} \\ \sqrt{\tilde{p} \cdot \sigma} \end{pmatrix} = \gamma^0 u(\tilde{p}) \quad \text{and} \quad v(p) = -\gamma^0 v(\tilde{p})$$

$$P\psi(x)P = \int \frac{d^3\tilde{p}}{(2\pi)^3} \underbrace{\frac{1}{\sqrt{2E_{\tilde{p}}}}}_{\text{invariant}} \sum_s \left( \eta_a a_{\vec{p}}^s \gamma^0 u(\tilde{p}) e^{-i\tilde{p}\tilde{x}} - \eta_b^* b_{\vec{p}}^s \gamma^0 v(\tilde{p}) e^{i\tilde{p}\tilde{x}} \right)$$

Let  $\eta_b^* = -\eta_a$ , so  $\eta_a = 1$  and  $\eta_b = -1$ .

We also require  $P\gamma^\mu P = \gamma^\mu$ . Need to verify that all observables are transformed as they should.

$$P\psi(x)P = \gamma^0 \psi(\tilde{x})$$

$$P\bar{\psi}(x)P = P\psi^\dagger(x)P \gamma^0 P = (P\psi(x)P)^\dagger \gamma^0 = \psi^\dagger(\tilde{x}) \gamma^0 \gamma^0 = \bar{\psi}(\tilde{x}) \gamma^0$$

- $P \bar{\psi}(x) \psi(x) P = \bar{\psi}(x) \psi(x)$

- $P \bar{\psi} \gamma^\mu \psi P = \bar{\psi}(x) \gamma^0 \gamma^\mu \gamma^0 \psi(x)$   

$$= \begin{cases} \bar{\psi}(x) \gamma^0 \psi(x), & \mu=0 \\ -\bar{\psi}(x) \gamma^i \psi(x), & \mu=i \end{cases}$$

- $P \bar{\psi} \gamma^\mu \gamma^5 \psi P = \bar{\psi}(x) \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \psi(x)$   

$$= \begin{cases} -\bar{\psi}(x) \gamma^0 \gamma^5 \psi(x) & \text{for } \mu=0 \\ +\bar{\psi}(x) \gamma^i \gamma^5 \psi(x) & \text{for } \mu=i \end{cases}$$

so  $\bar{\psi} \gamma^\mu \gamma^5 \psi$  is a pseudo-vector.

- $P \bar{\psi} \gamma^5 \psi P = -\bar{\psi}(x) \psi(x)$  is a pseudo-scalar.

$$P \bar{\psi} \gamma^\mu \frac{1 \pm \gamma^5}{2} \psi P = \begin{cases} \bar{\psi} \gamma^0 \frac{1 \mp \gamma^5}{2} \psi, & \mu=0 \\ -\bar{\psi} \gamma^i \frac{1 \mp \gamma^5}{2} \psi, & \mu=i \end{cases}$$

helicity  $\hat{p} \cdot \vec{S}$  flipped  
by  $P$  ( $\vec{p}$  is flipped, not  $\vec{S}$ )

$$P \bar{\psi} \frac{1 \pm \gamma^5}{2} \psi P = \bar{\psi} \frac{1 \mp \gamma^5}{2} \psi$$

## Summary of C, P, T

$$(-1)^n = \begin{cases} +1, & \mu=0 \\ -1, & \mu=i \end{cases}$$

|     | $\bar{\psi} \psi$ | $i \bar{\psi} \gamma^5 \psi$ | $\bar{\psi} \gamma^\mu \psi$ | $\bar{\psi} \gamma^\mu \gamma^5 \psi$ | $\partial_\mu$ |
|-----|-------------------|------------------------------|------------------------------|---------------------------------------|----------------|
| P   | +1                | -1                           | $(-1)^\mu$                   | $-(-1)^\mu$                           | $(-1)^\mu$     |
| T   | +1                | -1                           | $(-1)^\mu$                   | $(-1)^\mu$                            | $-(-1)^\mu$    |
| C   | +1                | +1                           | -1                           | +1                                    | +1             |
| CPT | +1                | +1                           | -1                           | -1                                    | -1             |