

Interacting fields

1) ϕ^4 -theory: $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$

- Simplest (self-)interacting theory ("interaction" between Fourier-components / plane waves)

! • Contains all essential features of an interacting theory, while being rel. simple.

- Not possible to solve (finding eigenstates and -values)
- Relevance to the standard model (self-int. of Higgs field) and in statistical mech. / condensed matter physics.

2) Yukawa-theory: $\mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Klein-Gordon}} - g \bar{\psi} \psi \phi$

- Very similar to QED while being simpler (no complications due to gauge inv. and polarization).
- Relevance to the standard model (coupling the Higgs field to quarks and leptons).

3) QED: $\mathcal{L} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{Dirac}} - e \underbrace{\bar{\psi} \gamma^\mu \psi A_\mu}_{j^\mu A_\mu = \mathcal{L}_{\text{int}}}$

$$\mathcal{L}_{\text{current density}}$$

$$= \mathcal{L}_{\text{Maxwell}} + \bar{\psi} (i \not{\partial} - m) \psi - e \bar{\psi} \not{A} \psi$$

$$= \mathcal{L}_{\text{Maxwell}} + \bar{\psi} (i \not{D} - m) \psi,$$

Gauge covariant derivative $D = \not{\partial} + i e \not{A}$, $D_\mu = \partial_\mu + i e A_\mu$

Explains nearly all observed phenomena from macroscopic scales down to $\sim 10^{-15}$ m.

The QED Lagrangian is invariant under gauge transf.:

$$A_\mu \mapsto A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

$$\psi \mapsto e^{i \alpha(x)} \psi$$

Correlation functions

Two-point correlation function $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$

$|\Omega\rangle$ = ground state / vac. state of the interacting / full theory
Used in QFT-calculations to obtain the S-matrix.

In a free theory the two-point corr. func. is the Feynman-prop.

Want to calculate the corr. func. Exact calc. not possible.

Interaction picture

Measurable quantities are expectation values $\langle \psi | A | \psi \rangle$

Define t_0 as a reference time when all pictures coincide.

Schrödinger picture: $|\psi(t)\rangle = e^{-iH(t-t_0)} |\psi(t_0)\rangle$, $A = A(t_0)$

Heisenberg picture : $|\psi\rangle = |\psi(t_0)\rangle$, $A(t) = e^{iH(t-t_0)} A(t_0) e^{-iH(t-t_0)}$

Interaction picture :

Write $H = H_0 + H_{int}$

$\uparrow$ interaction / small term

$$A_I(t) = e^{iH_0(t-t_0)} A(t_0) e^{-iH_0(t-t_0)} \quad \begin{array}{l} \text{time-ev. due to } H_0, \\ \text{free-field ev.} \end{array}$$

$$|\psi_I(t)\rangle = e^{iH_0(t-t_0)} |\psi(t)\rangle \\ = \underbrace{e^{iH_0(t-t_0)} e^{-iH(t-t_0)}}_{U(t,t_0)} |\psi(t_0)\rangle$$

$U(t,t_0)$ time-evolution op. in interaction pict.
from t_0 to t .

What is $U(t,t')$? (Not $e^{iH_0(t-t')} e^{-iH(t-t')}$ since t_0 is special:
 t_0 is the ref. time.)

$$\begin{aligned} \text{Must have } U(t_3, t_2) U(t_2, t_1) &= U(t_3, t_1), \text{ and } U(t_2, t_1) = U(t_1, t_2)^\dagger \\ U(t, t') &= U(t, t_0) U(t_0, t') = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} e^{iH(t'-t_0)} e^{-iH_0(t'-t_0)} \\ &= e^{iH_0(t-t_0)} e^{iH(t'-t)} e^{-iH_0(t'-t_0)} \end{aligned}$$

Hamiltonian in the interaction pict.:

$$e^{iH_0(t-t_0)} (H_0 + H_{\text{int}}) e^{-iH_0(t-t_0)} = H_0 + \underbrace{e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)}}_{H_I(t)}$$

Perturbation theory

Goal: Calculate $\langle \Omega | T \varphi(x) \varphi(y) | \Omega \rangle$ (corr. func., building block)

in φ^4 -theory $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4$.

We have $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}$ as for free KG field,

and $H = H_0 + \underbrace{\int d^3x \frac{\lambda}{4!} \varphi^4}_{\text{Hint}}$, where $H_0 = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} m^2 \varphi^2 \right]$.

Go to interaction picture:

$$\varphi_I(x) = e^{iH_0(t-t_0)} \underbrace{\varphi(t_0, \vec{x})}_{\text{field at ref. time to (any picture)}} e^{-iH_0(t-t_0)}, \quad \pi_I(x) = e^{iH_0(t-t_0)} \pi(t_0, \vec{x}) e^{-iH_0(t-t_0)} \quad (*)$$

$$\dot{\varphi}_I(x) = i H_0 \varphi_I(x) - i \varphi_I H_0 = i [H_0, \varphi_I(x)] \Rightarrow i \dot{\varphi}_I(x) = [\varphi_I(x), H_0]$$

$$i \dot{\pi}_I(x) = [\pi_I(x), H_0]$$

Since φ_I and π_I satisfy the Hamilton's eq. of motion with H_0 , we can use Probl. set 2, 3c) to deduce that $\varphi_I(x)$ satisfies the KG eq. of motion.

Solution:

$$\varphi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ipx} + a_p^\dagger e^{ipx})$$

$$\pi_I(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{E_p} (a_p e^{-ipx} - a_p^\dagger e^{ipx})$$

[Here a_p and $a_p^\dagger$ must first be interpreted as general expansion (operator) coefficients.]

(Commutator rel. in interact. p. (equal time):

$$[\varphi_I(\vec{x}, t), \pi_I(\vec{y}, t)] \stackrel{(*)}{=} i \delta(\vec{x} - \vec{y})$$

Thus the interaction-pict. fields behave exactly as the free fields.

So $a_p, a_p^\dagger$ are the usual ladder operators.

[$H_0 = H_{0I}$ so H_0 can be expressed with φ_I and π_I .
Leads to the usual expr. for H_0 with ladder op.
These ladder op. must be equal to those in the free theory, since the eigenstates are non-degenerate.]

The Heisenberg-p. field $\varphi(x)$ is determined from $\varphi_I(x)$ with

$$\begin{aligned} \varphi_H(x) &= e^{iH_0(t-t_0)} \varphi(\bar{x}, t_0) e^{-iH_0(t-t_0)}, & \varphi(x) &= e^{iH(t-t_0)} \varphi(\bar{x}, t_0) e^{-iH(t-t_0)} \\ \Rightarrow \varphi(x) &= e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \varphi_I(x) \underbrace{e^{iH_0(t-t_0)} e^{-iH(t-t_0)}}_{U(t, t_0)} \end{aligned}$$

$$= U^\dagger(t, t_0) \varphi_I(x) U(t, t_0) \quad (**)$$

↖ known

Need $U(t, t') = e^{iH_0(t-t_0)} e^{iH(t'-t)} e^{-iH_0(t'-t_0)}$ in terms of $H_I(t)$,

where $H_I(t) = e^{iH_0(t-t_0)} H_{int} e^{-iH_0(t-t_0)}$ is the interact. pict. version of H_{int} . ($H = H_0 + H_{int}$)

$$= e^{iH_0(t-t_0)} \int d^3x \frac{\lambda}{4!} \varphi^4(x) e^{-iH_0(t-t_0)}$$

$$= \int d^3x \frac{\lambda}{4!} \varphi_I^4(x).$$

↖ these are known!

$$\left[U_0^\dagger \varphi \varphi U_0 = U_0^\dagger \varphi U_0 U_0^\dagger \varphi U_0 = \varphi_I \varphi_I \right]$$

$$i \frac{\partial U(t, t')}{\partial t} = e^{iH_0(t-t_0)} (H - H_0) e^{iH(t'-t)} e^{-iH_0(t'-t_0)}$$

$$= e^{iH_0(t-t_0)} H_{int} e^{iH(t'-t)} e^{-iH_0(t'-t_0)}$$

$$= e^{iH_0(t-t_0)} H_{int} e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)} e^{iH(t'-t)} e^{-iH_0(t'-t_0)}$$

$$= H_I(t) U(t, t') \quad (\text{note } H_I(t) \text{ dependent on } t)$$

Solution:

$$U(t, t') = 1 + (-i) \int_{t'}^t dt_1 H_I(t_1) + (-i)^2 \int_{t'}^t dt_1 \int_{t'}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \dots$$

already in time order

since

$$\frac{\partial U}{\partial t} = 0 + (-i) H_I(t) + (-i)^2 \int_{t'}^t dt_2 H_I(t) H_I(t_2) + \dots$$

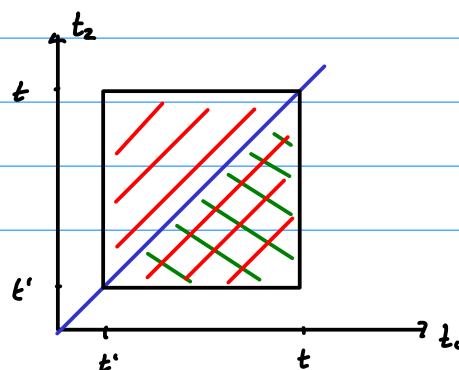
$$i \frac{\partial U}{\partial t} = H_I(t) + H_I(t) (-i) \int_{t'}^t dt_1 H_I(t_1) + \dots = H_I(t) U(t, t'), \quad \text{OK}$$

Useful rewrite:

$$U(t, t') = 1 + (-i) \int_{t'}^t dt_1 H_I(t_1) + \frac{(-i)^2}{2!} \int_{t'}^t \int_{t'}^{t_1} dt_1 dt_2 T \{ H_I(t_1) H_I(t_2) \} + \dots \quad (***)$$

$$= T \left\{ \exp \left[-i \int_{t'}^t dt'' H_I(t'') \right] \right\}$$

number of ways to order the factors



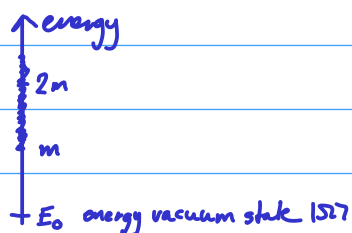
Dyson series.

With (**) and (***) we have a method of calculating $\varphi(x)$.

Need $|\Omega\rangle$.

Assume that the theory has a mass gap:

(ok for e.g. φ^4 -theory).



$$H = E_0 |\Omega\rangle\langle\Omega| + \int d\lambda \int \frac{d^3p}{(2\pi)^3} E_{\lambda\vec{p}} |\lambda\vec{p}\rangle\langle\lambda\vec{p}|, \quad E_{\lambda\vec{p}} \geq m.$$

$$\text{Claim: } \langle\Omega| T \varphi(x) \varphi(y) |\Omega\rangle = \frac{\langle 0| T \varphi_I(x) \varphi_I(y) U(\infty, -\infty) |0\rangle}{\langle 0| U(\infty, -\infty) |0\rangle} \quad (\text{Tang p. 76})$$

Assume $x^0 > y^0$.

$$\begin{aligned} \text{Numerator} &= \langle 0| U(\infty, x^0) \varphi_I(x) U(x^0, y^0) \varphi_I(y) U(y^0, -\infty) |0\rangle \\ &= \langle 0| U(\infty, x^0) U(x^0, t_0) \varphi(x) U^\dagger(x^0, t_0) U(x^0, y^0) U(y^0, t_0) \varphi(y) U^\dagger(y^0, t_0) U(y^0, -\infty) |0\rangle \\ &= \langle 0| U(\infty, t_0) \varphi(x) \varphi(y) U(t_0, -\infty) |0\rangle \end{aligned}$$

$$\text{Consider } \langle\psi| U(t_0, t') |0\rangle = \langle\psi| e^{iH(t'-t_0)} |0\rangle \quad (\text{we'll let } t' \rightarrow -\infty \text{ in the end})$$

↑
any state

$$U(t_0, t') = e^{iH(t'-t_0)} e^{-iH_0(t'-t_0)}, \quad H_0 = \int \frac{d^3p}{(2\pi)^3} E_p a_p^\dagger a_p$$

Insert a complete set of states:

$$\begin{aligned} &= \langle\psi| e^{iH(t'-t_0)} [|\Omega\rangle\langle\Omega| + \int d\lambda \int \frac{d^3p}{(2\pi)^3} |\lambda\vec{p}\rangle\langle\lambda\vec{p}|] |0\rangle \\ &= e^{iE_0(t'-t_0)} \langle\psi|\Omega\rangle\langle\Omega|0\rangle + \int d\lambda \int \frac{d^3p}{(2\pi)^3} e^{iE_{\lambda\vec{p}}(t'-t_0)} \langle\psi|\lambda\vec{p}\rangle\langle\lambda\vec{p}|0\rangle \\ &\quad \xrightarrow[t' \rightarrow -\infty]{} 0 \text{ by Riemann-Lebesgue lemma} \end{aligned}$$

$$\begin{aligned} \text{Numerator} &= \langle 0| U(t'', t_0) \varphi(x) \varphi(y) |\Omega\rangle\langle\Omega|0\rangle e^{iE_0(t'-t_0)} \\ &\xrightarrow[t'' \rightarrow \infty]{} e^{iE_0(t'-t_0)} e^{-iE_0(t''-t_0)} \langle 0|\Omega\rangle\langle\Omega|\varphi(x) \varphi(y) |\Omega\rangle\langle\Omega|0\rangle \end{aligned}$$

$$\text{Denominator} = \langle 0| U(t'', t') |0\rangle = \text{--- " --- except } \varphi(x) \varphi(y)$$

$$\frac{\text{Numerator}}{\text{Denominator}} = \frac{\langle\Omega|\varphi(x) \varphi(y) |\Omega\rangle}{\langle\Omega|\Omega\rangle} = \langle\Omega|\varphi(x) \varphi(y) |\Omega\rangle. \quad \text{OK.}$$

[Assume that $\langle\Omega|0\rangle \neq 0$ (intuitively ok if H_I is a suff. small perturbation), but perhaps not necc. since the factor is in both num. and denom.]

Conclusion: $\langle \Omega | T \varphi(x) \varphi(y) | \Omega \rangle = \frac{\langle 0 | T \varphi_I(x) \varphi_I(y) U(\infty, -\infty) | 0 \rangle}{\langle 0 | U(\infty, -\infty) | 0 \rangle}$

Interact. pict.:

$$= \frac{\langle 0 | T \varphi_I(x) \varphi_I(y) \exp[-i \int_{-\infty}^{\infty} dt H_I(t)] | 0 \rangle}{\langle 0 | \exp[-i \int_{-\infty}^{\infty} dt H_I(t)] | 0 \rangle}$$

$$\varphi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ipx} + a_p^\dagger e^{ipx})$$

$$H_I(t) = e^{iH_0(t-t_0)} H_{int} e^{-iH_0(t-t_0)} \\ = \int d^3x \frac{\lambda}{4!} \varphi_I^4(x) \quad (= H_{int} \text{ in the int. pict.})$$

$$U(\infty, -\infty) = T \exp[-i \int_{-\infty}^{\infty} dt H_I(t)], \text{ exp defined through power series.} \\ U(\infty, t_0) U(t_0, -\infty) = T \exp[-i \int_{-\infty}^{t_0} dt H_I(t)] T \exp[-i \int_{t_0}^{\infty} dt H_I(t)] \\ = T \left\{ \exp[-i \int_{-\infty}^{t_0} dt H_I(t)] \exp[-i \int_{t_0}^{\infty} dt H_I(t)] \right\} \\ = T \exp[-i \int_{-\infty}^{\infty} dt H_I(t)]$$

Wick's theorem

We have reduced the problem of calc. corr. func. to that of evaluating expressions of the form:

$$\langle 0 | T \varphi(x_1) \varphi(x_2) \dots \varphi(x_n) | 0 \rangle$$

Note: φ now denotes φ_I ,
i.e., fields in the interaction pict.

Write $\varphi(x) = \varphi^+(x) + \varphi^-(x)$,

$$\varphi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ipx} \quad , \quad \varphi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger e^{ipx} = [\varphi^+(x)]^+ \\ \text{pos. freq. part (} e^{-i\omega t} \text{ is pos. freq.)} \quad \text{neg. freq. part}$$

$$\varphi^+(x) | 0 \rangle = 0 \quad \text{and} \quad \langle 0 | \varphi^-(x) = 0. \\ \text{not dagger!}$$

Assume $x^0 > y^0$, and consider

$$T \varphi(x) \varphi(y) = \varphi(x) \varphi(y)$$

$$= \varphi^+(x) \varphi^+(y) + \varphi^+(x) \varphi^-(y) + \varphi^-(x) \varphi^+(y) + \varphi^-(x) \varphi^-(y)$$

$$= \varphi^+(x) \varphi^+(y) + \varphi^-(y) \varphi^+(x) + [\varphi^+(x), \varphi^-(y)] + \varphi^-(x) \varphi^+(y) + \varphi^-(x) \varphi^-(y)$$

If $y^0 > x^0$:

$$T \varphi(x) \varphi(y) = \varphi^+(x) \varphi^+(y) + \varphi^-(x) \varphi^+(y) + [\varphi^+(y), \varphi^-(x)] + \varphi^-(y) \varphi^+(x) + \varphi^-(x) \varphi^-(y)$$

For both cases:

$$T \varphi(x) \varphi(y) = \mathcal{N} \{ \varphi(x) \varphi(y) \} + \begin{cases} [\varphi^+(x), \varphi^-(y)], & x^0 > y^0 \\ [\varphi^+(y), \varphi^-(x)], & y^0 > x^0 \end{cases}$$

Normal ordering: put the + freq. op. to the right

Normal ordering : $N\{a_p a_k^\dagger a_q\} = a_k^\dagger a_p a_q$ (Puts annihilation operators to the right.)

Normal ordering and time-ordering work linearly (by def.) :

$$T\{\alpha \phi(x)\phi(y) + \beta \phi(z)\phi(w)\} = \alpha T\phi(x)\phi(y) + \beta T\phi(z)\phi(w)$$

(and similarly for N).

Wick's theorem for two fields: contraction

$$T\phi(x)\phi(y) = N\{\phi(x)\phi(y) + \overline{\phi(x)\phi(y)}\}$$

where

$$\left\{ [\phi^+(x), \phi^-(y)], x^0 > y^0 \right\} \equiv \overline{\phi(x)\phi(y)} = \langle 0 | T\phi(x)\phi(y) | 0 \rangle = D_F(x-y)$$

$$\left\{ [\phi^+(y), \phi^-(x)], y^0 > x^0 \right\}$$

Wick's theorem (writing $\phi(x_i)$ as ϕ_i), four fields:

$$T\{\phi_1 \phi_2 \phi_3 \phi_4\} = N\left\{ \phi_1 \phi_2 \phi_3 \phi_4 + \overline{\phi_1 \phi_2} \phi_3 \phi_4 + \overline{\phi_1 \phi_3} \phi_2 \phi_4 + \overline{\phi_1 \phi_4} \phi_2 \phi_3 + \overline{\phi_2 \phi_3} \phi_1 \phi_4 + \overline{\phi_2 \phi_4} \phi_1 \phi_3 + \overline{\phi_3 \phi_4} \phi_1 \phi_2 + \overline{\phi_1 \phi_2} \overline{\phi_3 \phi_4} \right\}$$

Wick's theorem on fields (proof by induction):

$$T\{\phi_1 \dots \phi_m\} = N\{\phi_1 \phi_2 \dots \phi_m + \text{all possible contractions}\}$$

Note: $\langle 0 | T\{\phi_1 \dots \phi_m\} | 0 \rangle = \sum \text{fully contracted } \phi_1 \dots \phi_m$

$$\text{For ex.: } \langle 0 | T\{\phi_1 \phi_2 \phi_3 \phi_4\} | 0 \rangle = \overline{\phi_1 \phi_2} \overline{\phi_3 \phi_4} + \overline{\phi_1 \phi_3} \overline{\phi_2 \phi_4} + \overline{\phi_1 \phi_4} \overline{\phi_2 \phi_3}$$

$$= D_F(x_1 - x_2) D_F(x_3 - x_4) + D_F(x_1 - x_3) D_F(x_2 - x_4) + D_F(x_1 - x_4) D_F(x_2 - x_3)$$

"Paradox": $1 = N(1) = N([a, a^\dagger]) = N(aa^\dagger - a^\dagger a) = a^\dagger a - a^\dagger a = 0$

Crucial step in induction proof for Wick's theorem:

$$\begin{aligned}
 [\varphi_1^+, N(\varphi_2 \varphi_3 \varphi_4)] &= [\varphi_1^+, N(\varphi_2^+ \varphi_3^+ + \varphi_2^- \varphi_3^- + \varphi_3^- \varphi_2^+ + \varphi_2^- \varphi_3^-)(\varphi_4^+ + \varphi_4^-)] \\
 &= [\varphi_1^+, \varphi_2^+ \varphi_3^+ \varphi_4^+ + \varphi_2^- \varphi_3^+ \varphi_4^+ + \varphi_3^- \varphi_2^+ \varphi_4^+ + \varphi_2^- \varphi_3^- \varphi_4^+ \\
 &\quad + \varphi_4^- \varphi_2^+ \varphi_3^+ + \varphi_4^- \varphi_2^- \varphi_3^- + \varphi_4^- \varphi_3^- \varphi_2^+ + \varphi_4^- \varphi_2^- \varphi_3^-] \\
 &= [\varphi_1^+, \varphi_2^-] \varphi_3^+ \varphi_4^+ + [\varphi_1^+, \varphi_3^-] \varphi_2^+ \varphi_4^+ + [\varphi_1^+, \varphi_2^-] \varphi_3^- \varphi_4^+ + \varphi_2^- [\varphi_1^+, \varphi_3^-] \varphi_4^+ \\
 &\quad + [\varphi_1^+, \varphi_4^-] \varphi_2^+ \varphi_3^+ + [\varphi_1^+, \varphi_4^-] \varphi_2^- \varphi_3^- + \varphi_4^- [\varphi_1^+, \varphi_2^-] \varphi_3^+ + [\varphi_1^+, \varphi_4^-] \varphi_3^- \varphi_2^+ \\
 &\quad + \varphi_4^- [\varphi_1^+, \varphi_3^-] \varphi_2^+ + [\varphi_1^+, \varphi_4^-] \varphi_2^- \varphi_3^- + \varphi_4^- [\varphi_1^+, \varphi_2^-] \varphi_3^- + \varphi_4^- \varphi_2^- [\varphi_1^+, \varphi_3^-] \\
 &= [\varphi_1^+, \varphi_2^-] N \varphi_3 \varphi_4 + [\varphi_1^+, \varphi_3^-] N \varphi_2 \varphi_4 + [\varphi_1^+, \varphi_4^-] N \varphi_2 \varphi_3
 \end{aligned}$$

Note that the $[\ ,\]$ are taken out of $N\{\}$.

Corr. functions: $\langle \Omega | T \varphi(x) \varphi(y) | \Omega \rangle = \frac{\langle 0 | T \{ \varphi_1(x) \varphi_1(y) e^{-i \int dt H_I(t)} \} | 0 \rangle}{\langle 0 | T e^{-i \int dt H_I(t)} | 0 \rangle}$
 with $H_I(t) = \int d^3x \frac{\lambda}{4!} \varphi^4$

$\langle 0 | T \varphi_1(x_1) \varphi_1(x_2) \dots \varphi_1(x_n) | 0 \rangle = \sum \text{products of Feynman prop.}$
 Wick's Theorem

Ex: $\langle 0 | T \varphi_1 \varphi_2 \varphi_3 \varphi_4 | 0 \rangle = \overbrace{\varphi_1 \varphi_2 \varphi_3 \varphi_4} + \overbrace{\varphi_1 \varphi_2 \varphi_3} \varphi_4 + \overbrace{\varphi_1 \varphi_2 \varphi_3} \varphi_4$
 $= \begin{array}{c} 1 \text{---} 2 \\ 3 \text{---} 4 \end{array} + \begin{array}{c} 1 \\ | \\ 3 \end{array} \begin{array}{c} 2 \\ | \\ 4 \end{array} + \begin{array}{c} 1 \text{---} 3 \\ 2 \text{---} 4 \end{array}$

Ex: $\langle \Omega | T \varphi(x) \varphi(y) | \Omega \rangle = \frac{\langle 0 | T \{ \underbrace{\varphi(x) \varphi(y)}_{\text{Heisenb. fields}} + \underbrace{\varphi(x) \varphi(y)}_{\text{interact. pict. fields}} [-i \int dt H_I(t)] + \dots \} | 0 \rangle}{\langle 0 | T \{ 1 - i \int dt H_I(t) + \dots \} | 0 \rangle}$

Num. = $\langle 0 | T \varphi(x) \varphi(y) (-i) \int dt \int d^3z \frac{\lambda}{4!} \varphi^4 | 0 \rangle$
 $= \frac{-i\lambda}{4!} \int d^4z \langle 0 | T \varphi(x) \varphi(y) \varphi(z) \varphi(z) \varphi(z) \varphi(z) | 0 \rangle$

$\underbrace{\varphi(x) \varphi(y)} \underbrace{\varphi(z) \varphi(z)} \underbrace{\varphi(z) \varphi(z)}$ 3 ways

$\underbrace{\varphi(x) \varphi(y) \varphi(z) \varphi(z)} \underbrace{\varphi(z) \varphi(z)}$ 4 · 3 ways

Num = $3 \frac{(-i\lambda)}{4!} D_F(x-y) \int d^4z D_F(z-x) D_F(z-y)$
 $+ 12 \frac{(-i\lambda)}{4!} \int d^4z D_F(x-z) D_F(y-z) D_F(z-z)$

$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$

$D_F(z-z) = \infty$
 but can be reg.
 by a cutoff in p .
 Postpone the
 discussion of
 loops.

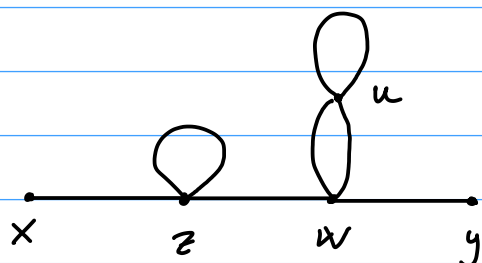
Feynman diagram:

$\begin{array}{c} x \text{---} y \\ \text{ } \end{array} \otimes z + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{ } \end{array}$

lines: propagators
 internal points: vertices

Next consider e.g. the expansion to 3th order:

$$\langle 0 | \phi(x) \phi(y) \frac{1}{3!} \left(\frac{-i\lambda}{4!} \right)^3 \int d^4z \phi \phi \phi \phi \int d^4u \phi \phi \phi \phi \int d^4u \phi \phi \phi \phi | 0 \rangle$$



We let this diagram represent all 10368 identical terms.

How many contr. give this same expr.?

$$3! \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot \frac{1}{2} = 10368$$

interchange of vertices placement of contr. into z vertex placement of contr. into w vertex placement of contr. into u vertex interchange of — and —.

In practice one usually draw the diagram first, then write down the corresponding expression.

What is the overall constant?

We have a (partial) cancellation of $\frac{3!}{3!}$ and $\left(\frac{4!}{4!}\right)^3$, so drop the factors $\frac{1}{3!} \mapsto 1$ and $\left(\frac{-i\lambda}{4!}\right) \mapsto (-i\lambda)$.

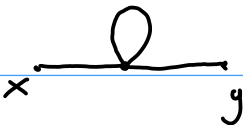
This gives a too large factor, so divide by the symmetry factor $S = 2 \cdot 2 \cdot 2 = 8$

$\uparrow$ line starting and ending at same vertex.
symmetric under interchange of the two ends of the loop.
=

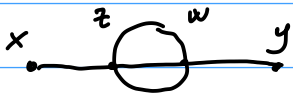
$\uparrow$
interchanging lines, propagators from w to u.

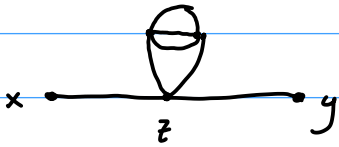
= symmetry changing directions of the loop

Third possibility: Equivalence of two vertices.
(then the $3!$ factor overcounts the number of contractions)

Examples:  $S = 2$ ^{orientation of loop}

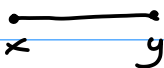
 $S = 2 \cdot 2 \cdot 2$ ^{two eq. lines}

 $S = 3!$ ^{3 lines connecting z and w}

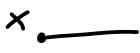
 $S = 3! \cdot 2$ ^{2 equiv. vertices}

Summary: $\langle \Omega | T \varphi(x) \varphi(y) | \Omega \rangle = \frac{\langle 0 | T \varphi_I(x) \varphi_I(y) e^{-i \int d^4x \mathcal{L}_I(x)} | 0 \rangle}{\langle 0 | T e^{-i \int d^4x \mathcal{L}_I(x)} | 0 \rangle}$
 $= \frac{\text{Num}}{\text{Den}} \cdot \text{Num} =$
 $= \text{sum of all possible diagrams with two external points}$

Translation between diagrams and analytic expr.:

1.  $= D_F(x-y)$ (propagator)

2.  $= (-i\lambda) \int d^4z$ (vertex)

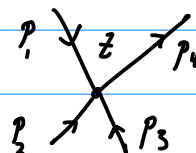
3.  $= 1$ (external points)

4. Divide by S .

Feynman rules! (In the position space.)

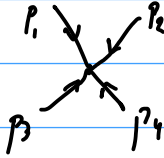
We have $D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$
^(why?)
 $= D_F(y-x)$

$(-i\lambda) \int d^4z e^{-ip_1^2 z - ip_2^2 z - ip_3^2 z + ip_4^2 z}$
 $= (-i\lambda) (2\pi)^4 \delta(p_1 + p_2 + p_3 - p_4)$ 4-momentum conservation



Momentum-space Feynman rules :

1. Propagator $\xrightarrow{p} = \frac{i}{p^2 - m^2 + i\epsilon}$

2. Vertex  $= (-i\lambda)$

3. External points $\overset{x}{\leftarrow} = e^{-ipx}$

4. Momentum conservation at each vertex

5. Integrate over each undetermined momentum $\int \frac{d^4 p}{(2\pi)^4}$

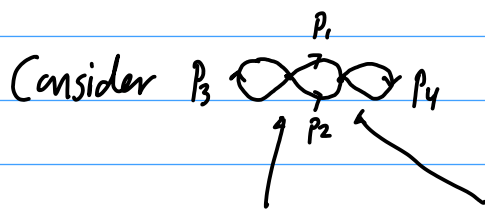
6. Divide by S .

For later: If we omit the e^{-ipx} for external points (in 3) and the associated $\int \frac{d^4 p}{(2\pi)^4}$ (in 5), we have Fourier-transformed the diagram to a momentum space expression as a function of external momenta p :

$$f(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \tilde{f}(p) \quad \text{for Fourier transform pair } f(x) \xleftrightarrow{F} \tilde{f}(p)$$

$$\langle \Omega | T \varphi(x) \varphi(y) | \Omega \rangle \xrightarrow{F} \int d^3 x e^{ipx} \int d^3 y e^{iqy} \langle \Omega | T \varphi(x) \varphi(y) | \Omega \rangle$$

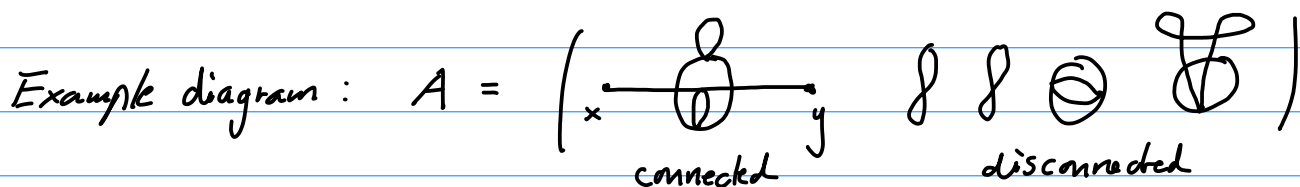
Exponentiation and cancellation



$$(2\pi)^4 \delta(p_1 + p_2) \quad (2\pi)^4 \delta(0) = \int d^4x e^{-i0x} = \text{volume spacetime}$$

(not nice...)

Let $V_i \in \{ \text{loop}, \text{tadpole}, \text{bubble}, \text{triangle}, \dots \} = \text{all different, disconnected from external points}$



$$A = (\text{value connected piece}) \cdot \prod_i \frac{1}{n_i!} V_i^{n_i}, \quad n_i = \# \text{ of } V_i$$

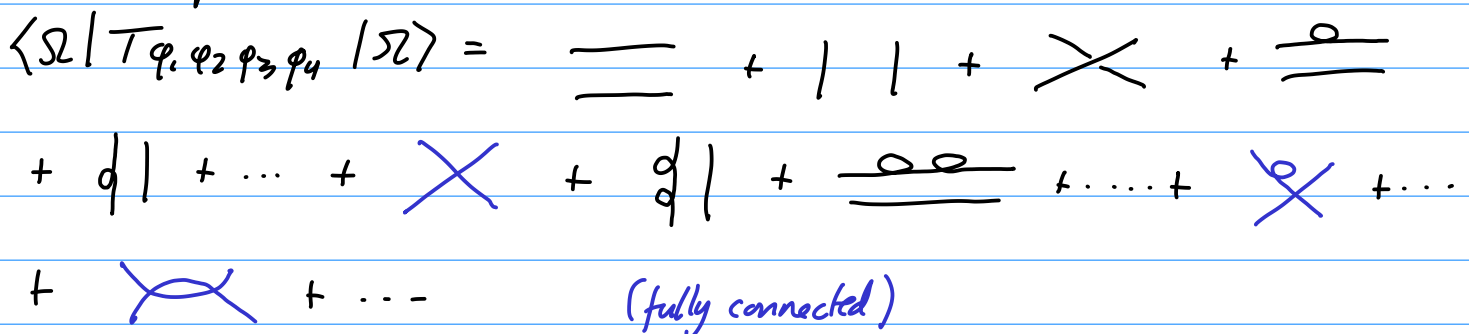
$$\text{Num} = \sum_{\text{all connected}} (\text{value conn.}) \cdot \sum_{n_1} \frac{1}{n_1!} V_1^{n_1} \sum_{n_2} \frac{1}{n_2!} V_2^{n_2} \dots$$

$$= \sum \text{connected} \cdot \exp(V_1 + V_2 + \dots)$$

$$\text{Denom} = \exp(V_1 + V_2 + \dots)$$

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \frac{\text{Num}}{\text{Denom}} = \sum \text{connected diagrams with two external points}$$

For example:

$$\langle \Omega | T \phi_1 \phi_2 \phi_3 \phi_4 | \Omega \rangle = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots$$


(fully connected)