

$$\underline{e^+ e^- \rightarrow \mu^+ \mu^-}$$

$$iM = \begin{array}{c} e^- \\ \nearrow p \\ \text{---} q \text{---} \\ \searrow k \\ \mu^- \\ \nearrow k' \\ \mu^+ \\ \nwarrow p' \\ e^+ \end{array} = \bar{v}^s(p') (-ie \gamma^\mu) u^r(p) \frac{-ig_{\mu\nu}}{q^2} \bar{u}^r(k) (-ie \gamma^\nu) v^{s'}(k')$$

$$= i \frac{e^2}{q^2} \bar{v}(p') \gamma^\mu u(p) \bar{u}(k) \gamma_\mu v(k'), \quad q = p + p' = k + k'$$

$$|M|^2 = \frac{e^4}{q^4} \left(\bar{v}(p') \gamma^\mu u(p) \bar{u}(p) \gamma^\nu v(p') \right) \left(\bar{u}(k) \gamma_\mu v(k') \bar{v}(k') \gamma_\nu u(k) \right)$$

↑ Problem 10: $(\bar{v} \gamma^\mu u)^\dagger = \bar{u} \gamma^\mu v$

We are interested in $\frac{1}{2} \sum_s \frac{1}{2} \sum_{s'} \sum_r \sum_{r'} |M(s, s' \rightarrow r, r')|^2$
 since incident beams usually are unpolarized (containing equal amounts of both spins)
 and detectors are insensitive to spins.

$$\frac{1}{4} \sum_{r, s, s'} |M|^2 = \frac{e^4}{4q^4} \text{tr}[(p' - m_e) \gamma^\mu (p + m_e) \gamma^\nu] \text{tr}[(k + m_\mu) \gamma_\mu (k' - m_\mu) \gamma_\nu]$$

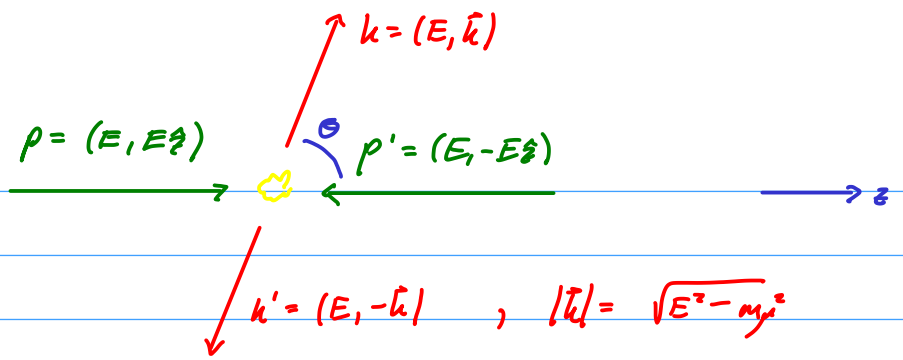
$$\begin{aligned} \text{tr}[(p' - m_e) \gamma^\mu (p + m_e) \gamma^\nu] &= -m_e^2 \text{tr} \gamma^\mu \gamma^\nu + \text{tr}(p' \gamma^\mu p \gamma^\nu) \\ &\quad \uparrow \text{tr}(\text{odd number of } \gamma\text{-matr}) = 0 \\ &= -4m_e^2 g^{\mu\nu} + p'_\kappa p_\lambda 4(g^{\kappa\mu} g^{\lambda\nu} - g^{\kappa\lambda} g^{\mu\nu} + g^{\kappa\nu} g^{\mu\lambda}) \\ &\quad \uparrow \text{tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}, \\ &\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\ &\text{tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho) = 4(g^{\mu\nu} g^{\lambda\rho} - g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\nu\lambda}) \\ &= 4[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu} (p' \cdot p + m_e^2)] \end{aligned}$$

$$\text{tr}[(k + m_\mu) \gamma_\mu (k' - m_\mu) \gamma_\nu] = 4[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} (k \cdot k' + m_\mu^2)]$$

Set $m_e = 0$ ($m_e/m_\mu \approx 1/200$).

$$\begin{aligned} \frac{1}{4} \sum_{\text{spins}} |M|^2 &= 4 \frac{e^4}{q^4} [2(p' \cdot k)(p \cdot k') + 2(p \cdot k)(p' \cdot k') - 2p \cdot p' (k \cdot k' + m_\mu^2) - 2k \cdot k' (p \cdot p')] \\ &\quad + 4(p' \cdot p)(k \cdot k') + 4m_\mu^2 p \cdot p'] \\ &= 8 \frac{e^4}{q^4} [(p \cdot k)(p' \cdot k') + (p \cdot k')(p' \cdot k) + m_\mu^2 p \cdot p'] \end{aligned}$$

CM frame.



$$q^2 = (p + p')^2 = (2E)^2 = 4E^2$$

$$p \cdot p' = E^2 + E^2 = 2E^2$$

$$p \cdot k = p' \cdot k' = E^2 - E|\hat{k}| \cos \theta$$

$$p \cdot k' = p' \cdot k = E^2 + E|\hat{k}| \cos \theta$$

$$\frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 = 8 \frac{e^4}{16E^4} \left[E^2 (E - |\hat{k}| \cos \theta)^2 + E^2 (E + |\hat{k}| \cos \theta)^2 + 2m_\mu^2 E^2 \right]$$

$$= \frac{e^4}{E^2} \left[E^2 + |\hat{k}|^2 \cos^2 \theta + m_\mu^2 \right] = \frac{e^4}{E^2} \left[E^2 + (E^2 - m_\mu^2) \cos^2 \theta + m_\mu^2 \right]$$

$$= e^4 \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2 \theta \right]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E 2E 2v} \frac{|\hat{k}|}{(2\pi)^2 4E_{cm}} \frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 = \frac{1}{2E_{cm}^2} \frac{|\hat{k}|}{16\pi^2 E_{cm}} \frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2$$

$\downarrow$ velocity of one of e in CM.
 $\frac{|\hat{p}|}{E} = v \approx 1$

$\downarrow$ $2E$

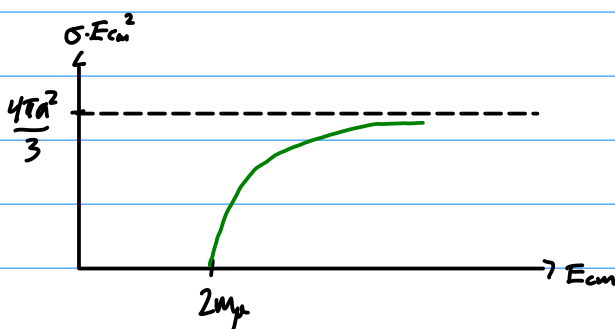
$$= \frac{\alpha^2}{4E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2 \theta \right]$$

$\alpha = \frac{e^2}{4\pi}, \alpha^2 = \frac{e^4}{16\pi^2}$

$$\int d\Omega \cos^2 \theta = 2\pi \int_0^\pi d\theta \sin \theta \cos^2 \theta = 2\pi \int_{-1}^1 d \cos \theta \cos^2 \theta = \frac{4\pi}{3}$$

$$\sigma = 4\pi \frac{\alpha^2}{4E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[1 + \frac{m_\mu^2}{E^2} + \frac{1}{3} \left(1 - \frac{m_\mu^2}{E^2}\right) \right]$$

$$= \frac{4\pi\alpha^2}{3E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right]$$



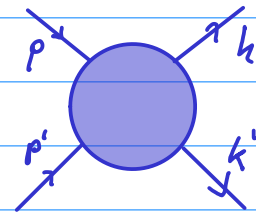
Method QFT lowest order

1. Draw the diagram(s) for the desired process.
2. Feynman rules $\Rightarrow i\mathcal{M}$
3. $\sum_{\text{spins}} |\mathcal{M}|^2$ using $\sum_s u^s(p) \bar{u}^s(p) = \not{p} + m$, $\sum_s v^s(p) \bar{v}^s(p) = \not{p} - m$
4. Evaluate traces using trace theorems
5. Choose frame (CM?), figure, express kinematic variables in terms of $E, \theta, \dots$. Alternatively Lorentz inv. in terms of Mandelstam-v.
6. Plug into formula for $d\sigma$.

Mandelstam variables

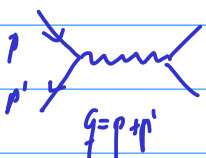
Define $s = (p + p')^2 = (k + k')^2$
 $t = (k - p)^2 = (k' - p')^2$
 $u = (k' - p)^2 = (k - p')^2$

Lorentz-invariant variables.

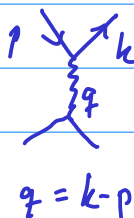


Ex.: $e^+e^- \rightarrow \mu^+\mu^-$. $\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4} [(p \cdot k)(p' \cdot k') + (p \cdot k')(p' \cdot k) + m_\mu^2 (p \cdot p')]$
 $= \frac{8e^4}{s^2} \left[\left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2 \right]$
 $\swarrow$ ultrarelativistic case, $m_\mu \approx 0$
 $t = -2kp = -2k'p'$, $u = -2k'p = -2kp'$

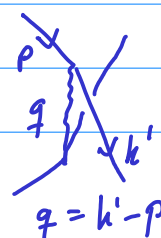
s-channel:



t-channel:



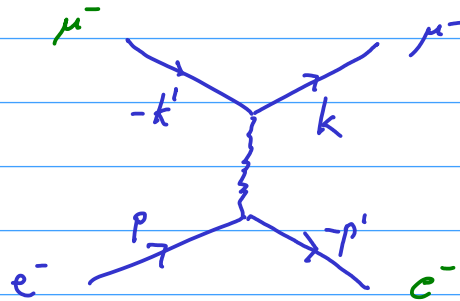
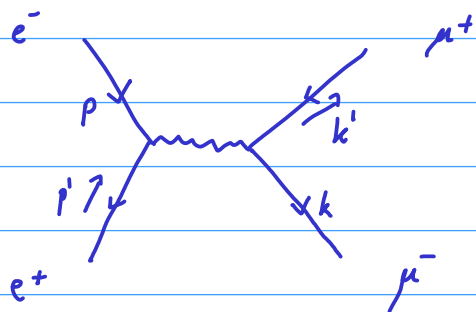
u-channel:



Crossing symmetry

$$e^+e^- \rightarrow \mu^+\mu^-$$

$$e^-\mu^- \rightarrow e^-\mu^-$$



Symmetry:

incoming $e^+ \mapsto$ outgoing e^-

outgoing $\mu^+ \mapsto$ incoming μ^-

Same value of Feynman diagram except possibly a difference due to external leg factors.

$$\text{When } e^+ \xrightarrow{\vec{p}'} \quad \mapsto \quad \xrightarrow{-\vec{p}'} e^-$$

$$\bar{v}(p') \quad \mapsto \quad \bar{u}(-p')$$

$$\text{In } \sum_{\text{spins}} |\mathcal{M}|^2 \text{ we then get } v(p') \bar{v}(p') \mapsto u(-p') \bar{u}(-p')$$

$$p' - m \mapsto -p' + m = -(p' - m)$$

Same, except for a minus for each crossed fermion.

$$\text{Original Mandelstam v. } s, t, u \mapsto s', t', u'$$

$$s = (p + p')^2 = (p - (-p'))^2 = t'$$

$$t = (k - p)^2 = u'$$

$$u = (k' - p)^2 = (p - k')^2 = s'$$

$$e^+e^- \rightarrow \mu^+\mu^- : \quad \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{s^2} \left[\left(\frac{t}{2} \right)^2 + \left(\frac{u}{2} \right)^2 \right]$$

$\Downarrow$ crossing symmetry

$$e^-\mu^- \rightarrow e^-\mu^- : \quad \frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{t^2} \left[\left(\frac{u}{2} \right)^2 + \left(\frac{s}{2} \right)^2 \right]$$

Photon polarization sums

Photon polarization vectors:

Orient coord. system s.t. $h^\mu = (h, 0, 0, h)^T$, $\varepsilon_0^\mu = (1, 0, 0, 0)^T$
 $\varepsilon_1^\mu = (0, 1, 0, 0)^T$, $\varepsilon_3^\mu = (0, 0, 0, 1)^T$
 $\varepsilon_2^\mu = (0, 0, 1, 0)^T$

$$\sum_{r=1}^2 \varepsilon_r \varepsilon_r^\dagger = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} = \underbrace{-g^{\mu\nu}}_{\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}} - \underbrace{\frac{k^\mu k^\nu}{k}}_{\begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & & & \\ 0 & & & \\ -1 & 0 & 0 & -1 \end{pmatrix}} + \underbrace{\frac{k^\mu}{k} \varepsilon_0^\nu}_{\begin{pmatrix} 1 & & & \\ 0 & & & \\ 0 & & & \\ 1 & & & \end{pmatrix}} + \underbrace{\varepsilon_0^\mu \frac{k^\nu}{k}}_{\begin{pmatrix} & 1 & 0 & 0 & 1 \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}}$$

Consider $\tilde{\chi}_{i,r}^{k^*,r}$  $= i\mathcal{M} = i\mathcal{M}_\mu(k) \varepsilon_r^{k^*}(k)$
 $\downarrow$ $\mathcal{M}$ -matrix except $\varepsilon_r^{k^*}(k)$

Field external wave : $A^*(x) = \epsilon_r^m(k) e^{-ikx}$

Gauge transformation: $A^\mu(x) \mapsto A^\mu(x) + \partial^\mu f(x)$, let $f(x) = e^{-ikx}$
 $= [\mathcal{E}_r(k) - ik^\mu] e^{-ikx}$

so : $\varepsilon_r^\wedge(h) \mapsto \varepsilon_r^\wedge(h) - i h^\wedge$

To have iM invariant, we must have

$$k^\mu M_{\mu}(k) = 0 \quad \text{Ward's identity}$$

In calculating cross sections we need

$$\sum_r |M|^2 = M_\mu^*(k) M_\nu(k) \sum_{r=1}^2 \epsilon_r^\mu(k) \epsilon_r^\nu(k)$$

$$\stackrel{\text{Ward}}{=} M_\mu^*(k) M_\nu(k) (-g^{\mu\nu})$$

Thus in these calculations:

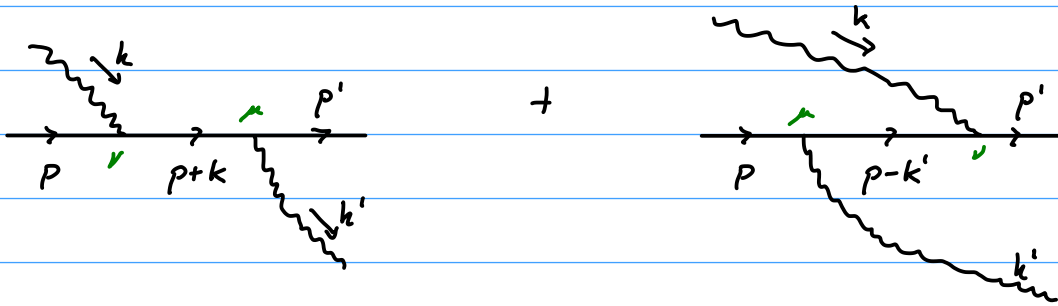
$$\sum \varepsilon_r^\mu(k) \varepsilon_\nu^\nu(k) \rightarrow -g^{\mu\nu}$$

Crossing symmetry photon:

$$\begin{array}{ccc} \text{---} \nearrow \text{---} & \mapsto & \text{---} \searrow \text{---} \\ \downarrow k & & \downarrow -k \\ \varepsilon_r(k) & \mapsto & \varepsilon_r^*(-k) \end{array}$$

$$\sum_r \xi_r^{\mu} (h) \xi_r^{\nu} (h) \mapsto \sum_r \xi_r^{\nu} (-h) \xi_r^{\mu} (-h)$$

Compton scattering $e^- \gamma \rightarrow e^- \gamma$



No relative minus sign since fermion parts of the diagram is the same; they differ only by an interchange of μ and ν .

$$i\mathcal{M} = (-ie)^2 \bar{u}(p') \gamma^\mu \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2} \gamma^\nu u(p) \epsilon_\mu^*(k') \epsilon_\nu(k) \\ + (-ie)^2 \bar{u}(p') \gamma^\nu \frac{i(\not{p} - \not{k}' + m)}{(p-k')^2 - m^2} \gamma^\mu u(p) \epsilon_\mu^*(k') \epsilon_\nu(k)$$

We have $p^2 = m^2 = p'^2$ and $k^2 = k'^2 = 0$ (why not $(p+k)^2 = m^2$?).

Denominators: $(p+k)^2 - m^2 = 2pk$, $(p-k')^2 - m^2 = -2pk'$

Numerators: $(p+m) \gamma^\nu u(p) = \gamma^\nu (-\not{p} + m) u(p) + 2p^\nu u(p) \stackrel{\text{Dirac}}{=} 2p^\nu u(p)$

$$\int \not{p} \gamma^\nu = p_\mu \gamma^\mu \gamma^\nu = p_\mu (-\delta^{\mu\nu} + 2g^{\mu\nu}) = -\delta^{\mu\nu} p + 2p^\nu$$

$$i\mathcal{M} = -ie^2 \epsilon_\mu^*(k') \epsilon_\nu(k) \bar{u}(p') \left[\frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu}{2pk'} \right] u(p)$$

$$\frac{1}{4} \sum_{\text{pol. spin}} |\mathcal{M}|^2 = \frac{e^4}{4} g_{\mu\sigma} g_{\nu\alpha} \text{tr} \left\{ (\not{p}' + m) \left[\frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2pk} + \frac{\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu}{2pk'} \right] \right. \\ \left. \cdot (\not{p} + m) \left[\frac{\gamma^\sigma \not{k} \gamma^\rho + 2\gamma^\sigma p^\rho}{2pk} + \frac{\gamma^\rho \not{k}' \gamma^\sigma - 2\gamma^\rho p^\sigma}{2pk'} \right] \right\}$$

$\sum_{\text{pol. spin}} \epsilon_\mu^*(k') \epsilon_\sigma(k') \rightarrow (-g_{\mu\sigma})$

• remove terms with an odd # γ -matr.

$$\bullet \text{tr}(\not{p}' \gamma^\mu \not{k} \gamma^\nu \not{p} \gamma_\mu \not{k} \gamma_\nu) = \text{tr}((-2p') \not{k} (-2p) \not{k}) = 4 \text{tr}(\not{p}' \not{k} \not{p} \not{k}) \\ \stackrel{\int \gamma^\mu \not{p} \gamma_\mu = -2\not{p}}{=} 4 \text{tr}(\not{p}' \not{k} (2pk - \not{k} p))$$

$$= 8 \text{tr}(\not{p}' \not{k}) pk$$

$$= 32(pk)(p'k)$$

$$\int \text{tr} \not{p}' \not{k} = p'_\mu k_\nu \text{tr} \gamma^\mu \gamma^\nu = p'_\mu k_\nu \text{tr} \left(\frac{\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu}{2} \right) = p'_\mu k_\nu \text{tr} 1_{4 \times 4}$$

$$\int \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\not{p} \not{k} = p_\mu k_\nu \gamma^\mu \gamma^\nu$$

$$= p_\mu k_\nu (2g^{\mu\nu} - \gamma^\mu \gamma^\nu) = 2pk - \not{k} p$$

• Express in terms of Mandelstam var. s, t, u , use $s+t+u = 2m^2$

$$\Rightarrow \frac{1}{4} \sum_{\text{spins}} |M|^2 = 2e^4 \left[\frac{p \cdot k'}{p \cdot k} + \frac{p \cdot k}{p \cdot k'} + 2m^2 \left(\frac{1}{p \cdot k} - \frac{1}{p \cdot k'} \right) + m^4 \left(\frac{1}{p \cdot k} - \frac{1}{p \cdot k'} \right)^2 \right]$$

Choose frame: "Lab frame" where e^- is at rest.

Before:

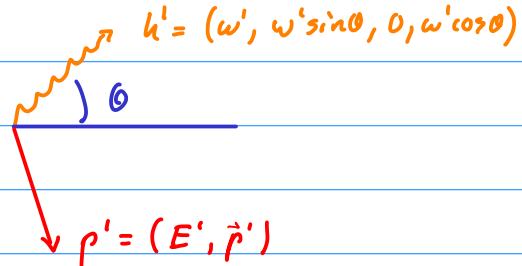


$$k = (\omega, \omega \hat{z})$$



$$p = (m, 0)$$

After:



What is ω' ? $m^2 = p'^2 = (p + k - k')^2$

$$= m^2 + 2pk - 2pk' - 2kk'$$

$$0 = m(\omega - \omega') - \omega\omega' + \omega\omega' \cos \theta$$

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m} (1 - \cos \theta)$$

Compton's formula for shift in photon wavelength.

Phase space integral $\int d\Pi_2 \dots$

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2} \left(\frac{\omega'}{\omega} \right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta \right], \text{ Klein-Nishina formula}$$

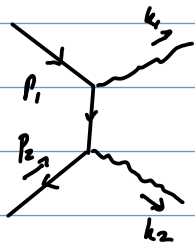
$$d\sigma = f(\varphi, \theta) d\Omega = f(\varphi, \theta) \sin\theta d\theta d\varphi$$

$$= 2\pi f(\theta) d\cos\theta$$

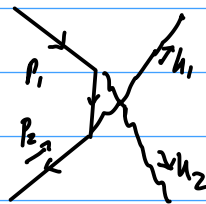
if $f(\theta)$ is independent on φ (rotational symmetry).

$$\text{since } \int_0^\pi \sin\theta d\theta = \int_{-1}^1 (-d\cos\theta) = \int_{-1}^1 d\cos\theta$$

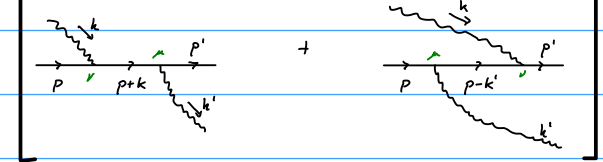
Pair annihilation into photons $e^+e^- \rightarrow 2\gamma$



+



Compton scatt:



Crossing symmetry: $p \mapsto p_1$ $k \mapsto -k_1$
 $p' \mapsto -p_2$ $k' \mapsto k_2$

Remember an extra -1 for $\sum |M|^2$ since there is one crossed fermion.

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$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s} \left(\frac{1+\cos^2\theta}{\sin^2\theta} \right) \text{ in high-energy limit.}$$