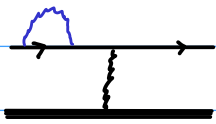


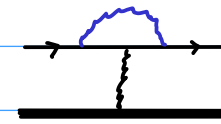
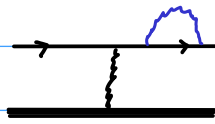
Radiative corrections



external leg corrections

renormalization: $m_0 \rightarrow m$

Done!



vertex correction

field strength renorm.

$\int$ heavy fermion



vacuum polarization

$e_0 \rightarrow e$

Diagrams with loops involving the heavy part. ignored, since $\frac{p+m}{p^2-m^2+i\epsilon}$ then is small.

Loops: integration of undetermined momenta.

Divergent integrals for large k ... ultraviolet divergences!

Why do these divergences happen? Cannot expect the theory to be valid for arb. small dimensions.

Solution: 1) Regularization (make them finite). E.g.: $\int_0^\infty dk \rightarrow \int_0^\Lambda dk$.
Then physical results are dependent on m_0 , e_0 and Λ .

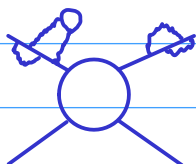
2) Renormalization:

None of these parameters are physically observable.

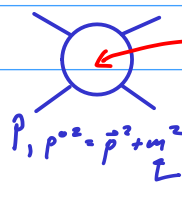
So choose m_0 or e_0 as functions of Λ such that observable quantities get dep. on new parameters e , m , but not Λ . For renormalizable theories this is possible, for all terms in pert. series, with only the two par. e and m !

Simplified ex.: Assume $M = e_0 \int_0^\infty dk C = \infty$. Cutoff: $M = e_0 \int_0^\Lambda dk C = e_0 \Lambda C$.

Let $e_0 = e_0(\Lambda) = \frac{e}{\Lambda}$. Then $M = e C$.



amputate

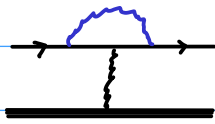


but the internal diagrams are still expressed with m_0 !

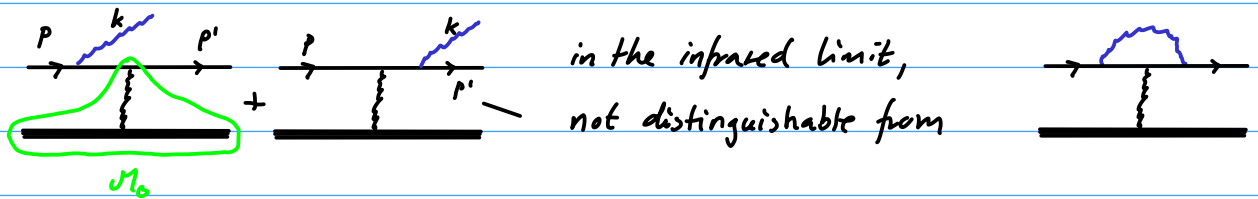
LSZ: $iM = \sqrt{Z}^4 \sum (\text{connected diag.})$

$p, p^2 = \vec{p}^2 + m^2$
 $\int$ physical mass

Vertex correction



- Problems:
- Ultraviolet divergence: Repaired by renormalization.
 - Infrared divergence: Cancelled by divergence for bremsstrahlung:



Bremsstrahlung: $iM = -ie \bar{u}(p') M_0(p', p-k) \frac{i(\not{p}-\not{k}+m)}{(p-k)^2 - m^2} \gamma^\mu \epsilon_\mu^*(k) u(p)$

+ ...

$$= \bar{u}(p') M_0(p', p) u(p) \cdot e \left[\frac{p' \cdot \epsilon^*}{p' \cdot k} - \frac{p \cdot \epsilon^*}{p \cdot k} \right]$$

1. $M_0(p', p-k) \approx M_0(p', p)$ since $|k| \ll |p' - p|$
 • same simplification as for Compton-scattering
 • ignore k

$$= \text{[Tree-level diagram]} \cdot e \left[\frac{p' \cdot \epsilon^*}{p' \cdot k} - \frac{p \cdot \epsilon^*}{p \cdot k} \right]$$

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \cdot |M(p_A, p_B \rightarrow \{p_f\})|^2 (2\pi)^4 \delta(p_A + p_B - \sum p_f)$$

$$d\sigma(p \rightarrow p' + r) = d\sigma(p \rightarrow p') \cdot \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2|k|} \sum_{\lambda=1,2} e^2 \left| \frac{p' \cdot \epsilon^\lambda}{p' \cdot k} - \frac{p \cdot \epsilon^\lambda}{p \cdot k} \right|^2 \quad k^\lambda = k(1, 0, 0, 1)$$

probability of radiating a photon with momentum in $d^3 k$ about $\vec{k}$, given that $p \rightarrow p'$ for the electron.

= ...

$$\sim d\sigma(p \rightarrow p') \cdot \frac{\alpha}{\pi} \log\left(\frac{-q^2}{\mu^2}\right) \log\left(\frac{-q^2}{m^2}\right), \quad -q^2 \text{ large}, \quad q = p' - p$$

$$q^2 = (E' - E)^2 - (\vec{p}' - \vec{p})^2$$

$$\approx -(\vec{p}' - \vec{p})^2 = -\vec{q}^2$$

μ : "photon mass".

Vertex correction



$$= i\mathcal{M} = ie^2 \bar{u}(p') T^\mu(p, p') u(p) \frac{1}{q^2} \bar{u}(k') \gamma_\mu u(k)$$

Lorentz-inv.: $\bar{u}(p') T^\mu u(p)$ is a vector.

$$T^\mu(p, p') = A \gamma^\mu + B (p'^\mu + p^\mu) + C (p'^\mu - p^\mu)$$

A, B, C can involve $e, m, p^2 = m^2 = p'^2, q^2 = -2p'p + 2m^2,$

$$pq = pp' - p^2, p'q = p'^2 - p'p, p, p', q.$$

A is a number.

$$\text{since } \not{p} u(p) = m u(p) \\ \bar{u}(p') \not{p}' = \bar{u}(p') m$$

Ward's identity: $0 = \bar{u}(p') T^\mu q_\mu u(p)$

$$\begin{aligned} &= A \bar{u}(p') \not{q} u(p) & 0 \\ &+ B \bar{u}(p') (p'^\mu + p^\mu) (p'_\mu - p_\mu) u(p) & 0 \\ &+ C \bar{u}(p') q^2 u(p) & \neq 0 \end{aligned}$$

not nec. zero, when $q^2 \neq 0$:

$$\begin{aligned} \bar{u}(p') u(p) &= (\xi_r^\dagger \sqrt{p' \cdot \sigma}, \xi_r^\dagger (p' \cdot \sigma)) \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ \sqrt{p \cdot \sigma} \xi_s \end{pmatrix} \\ &= \xi_r^\dagger (\sqrt{p' \cdot \sigma} \sqrt{p \cdot \sigma} + \sqrt{p' \cdot \sigma} \sqrt{p \cdot \sigma}) \xi_s \\ &\stackrel{\text{nonrel. limit}}{=} 2m \xi_r^\dagger \xi_s \end{aligned}$$

$$T^\mu(p, p') = A(q^2) \gamma^\mu + B(q^2) (p'^\mu + p^\mu) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)$$

$$\text{Gordon's identity: } \bar{u}(p') \frac{p'^\mu + p^\mu}{2m} u(p) = \bar{u}(p') \left[\gamma^\mu - \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p)$$

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$

$F_1(q^2)$ and $F_2(q^2)$: "form factors".

To lowest order: $F_1(q^2) = 1, F_2(q^2) = 0$. Will be computed to next order.

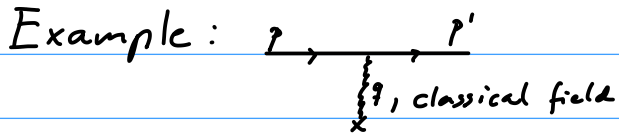
$$\Rightarrow F_1(q^2) = 1 + O(\alpha), F_2(q^2) = O(\alpha)$$

F_1 and F_2 describe the interaction between e^- and an e.m. field.

Feynman rule, int. w. classical field (pr. set 10):

$$i\mathcal{M} 2\pi\delta(p'^0 - p^0) = -ie \bar{u}(p') T^a(p', p) u(p) \cdot \tilde{A}_\mu(p' - p)$$

← classical field



a) $A_\mu(x) = (\phi(\vec{x}), 0)$ cl. static potential: interaction $e^- \leftrightarrow \phi(\vec{x})$ non-rel.

$$A_\mu(q) = (2\pi\delta(q^0) \tilde{\phi}(\vec{q}), 0)$$

$$i\mathcal{M} = -ie \bar{u}(p') T^0(p', p) u(p) \cdot \tilde{\phi}(\vec{q})$$

Assume $\phi(\vec{x})$ slowly varying; then $\tilde{\phi}(\vec{q})$ is concentrated about $\vec{q}=0$:

$$T^0(p', p) \approx \gamma^0 F_1(0)$$

$$\text{Nonrel. limit } \bar{u}(p') \gamma^0 u(p) \approx 2m \xi^{\dagger} \xi$$

$$i\mathcal{M} = -ie F_1(0) \tilde{\phi}(\vec{q}) 2m \xi^{\dagger} \xi$$

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \sigma} \xi \end{pmatrix}$$

$$\approx \sqrt{m} \begin{pmatrix} (1 - \frac{\vec{p} \cdot \vec{\sigma}}{2m}) \xi \\ (1 + \frac{\vec{p} \cdot \vec{\sigma}}{2m}) \xi \end{pmatrix}$$

$$\propto \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

As seen before: $i\mathcal{M} = -i\tilde{V}(\vec{q})$ (ignoring $2m \xi^{\dagger} \xi$)

← potential

$$\tilde{V}(\vec{q}) = e F_1(0) \tilde{\phi}(\vec{q})$$

$$V(\vec{x}) = e F_1(0) \phi(\vec{x})$$

Must have $F_1(0) = 1$.

b) Static $\vec{B}$ -field: $A_\mu(x) = (0, \vec{A}(\vec{x}))$, $\tilde{A}_\mu(q) = 2\pi\delta(q^0) (0, \tilde{\vec{A}}(\vec{q}))$.

$$i\mathcal{M} = ie \left[\bar{u}(p') \left(\gamma^i F_1 + \frac{i \sigma^{i\nu} q_\nu}{2m} F_2 \right) u(p) \right] \tilde{A}^i(\vec{q})$$

← raised, gives a -.

$$(\xi^{\dagger}, \xi^{\dagger}) \gamma^0 \gamma^i \begin{pmatrix} \xi \\ \xi \end{pmatrix} = 0 \text{ so must use a better approx.}$$

$$\text{Pr. set 7: } \bar{u}(p') \gamma^i u(p) = 2m \xi^{\dagger} \left(-\frac{i}{2m} \epsilon^{ijk} q_j \sigma^k \right) \xi + \text{term indep. of spin}$$

$$\bar{u}(p') \left(\frac{i}{2m} \sigma^{i\nu} q_\nu \right) u(p) = 2m \xi^{\dagger} \left(-\frac{i}{2m} \epsilon^{ijk} q_j \sigma^k \right) \xi, \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$

used $q^0=0$, $\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$, $u \approx \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$.

$$= -ie 2m \xi^{\dagger} \left(\frac{-1}{2m} \sigma^k [F_1(0) + F_2(0)] \right) \xi \tilde{B}^k(\vec{q}), \quad ij \mapsto ji$$

$$\tilde{B}^k(\vec{q}) = -i \epsilon^{ijk} q^j \tilde{A}^i(\vec{q}), \quad \vec{B}(\vec{x}) = \nabla \times \vec{A}(\vec{x}).$$

What does it mean that $q^0 \approx 0$?
 $x \approx 0$ makes no sense while $x \approx a$ does (it means $|x-a| \ll a$).
 Bandwidth in q : Δq
 $\Delta q \sim \frac{1}{\Delta x}$ ← typical dim. in $V(\vec{x})$

At least we must have $\Delta q \ll m$
 $\Delta x \gg m^{-1}$ Compton w.

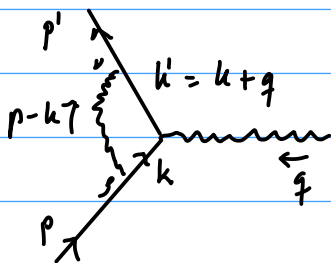
$$V(\vec{x}) = -\langle \vec{\mu} \rangle \cdot \vec{B}(x),$$

$$\begin{aligned} \langle \vec{\mu} \rangle &= \frac{e}{m} [F_1(0) + F_2(0)] \left\{ \vec{1} + \frac{\vec{\sigma}}{2} \right\} \\ &= g \left(\frac{e}{2m} \right) \vec{S}, \quad \vec{S} = \left\{ \vec{1} + \frac{\vec{\sigma}}{2} \right\} \end{aligned}$$

$$\text{Landé } g\text{-factor } g = 2 [F_1(0) + F_2(0)] = 2 + \underbrace{2 F_2(0)}.$$

anomalous mag. moment of the electron.

Evaluation of vertex correction



$$\bar{u}(p') \delta T^\mu(p', p) u(p) = \int \frac{d^4 k}{(2\pi)^4} \frac{-ig\gamma_\mu}{(k-p)^2 + i\epsilon}$$

$$\begin{aligned} &\cdot \bar{u}(p') (-ie\gamma^\nu) \frac{i(k' + m)}{k'^2 - m^2 + i\epsilon} \gamma^\mu \frac{i(k + m)}{k^2 - m^2 + i\epsilon} (-ie\gamma^\rho) u(p) \\ &= 2ie^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}(p') [k \gamma^\mu k' + m^2 \gamma^\mu - 2m(k + k')^\mu] u(p)}{(k-p)^2 + i\epsilon (k'^2 - m^2 + i\epsilon) (k^2 - m^2 + i\epsilon)} \quad (*) \end{aligned}$$

Numerator (ignoring $e^2, \bar{u}(p'), u(p)$):

$$-i\delta_\mu^\nu (k' + m) \gamma^\mu (k + m) \gamma^\nu = -i(-2m^2 \gamma^\mu - 2k \gamma^\mu k' + 4m k^\mu + 4m k'^\mu)$$

$$\gamma_\mu \gamma^\mu \gamma^\nu = -2\gamma^\nu$$

$$\gamma^\mu \gamma^\mu \gamma^\nu \gamma_\mu = 4\gamma^\nu$$

Numerator... Pr. set 12.

$$\gamma^\mu \gamma^\mu \gamma^\nu \gamma^\mu \gamma_\mu = -2\gamma^\nu \gamma^\mu \gamma^\mu$$

Feynman parameters

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} = \int_0^1 dx dy \delta(x+y-1) \frac{1}{[xA+yB]^2}$$

$$\frac{1}{B-A} \int_A^B \frac{dt}{t^2} = \frac{1}{B-A} \left[-\frac{1}{t} \right]_A^B = \frac{\frac{1}{A} - \frac{1}{B}}{B-A} \cdot AB = \frac{B-A}{(B-A)AB} = \frac{1}{AB}$$

$$\text{Put } t = B + (A-B)x. \quad \frac{1}{AB} = \frac{1}{B-A} \int_0^1 \frac{(A-B)dx}{[B + (A-B)x]^2} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}$$

Can be generalized by induction:

$$\frac{1}{A_1 A_2 \dots A_n} = \int_0^1 dx_1 \dots dx_n \delta(\sum x_i - 1) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \dots + x_n A_n]^n}$$

Denominator (k): $\frac{1}{((k-p)^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} = \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{D^3}$

$$D = x(k^2 - m^2 + i\epsilon) + y((k+q)^2 - m^2 + i\epsilon) + z((k-p)^2 + i\epsilon)$$

$$= k^2 - (x+y)m^2 + y(2kq + q^2) + z(-2kp + p^2) + i\epsilon$$

$\downarrow x+y+z=1$

$$D = k^2 + 2k(yq - zp) + yq^2 + zp^2 - (x+y)m^2 + i\epsilon$$

Define a shifted k: $\ell = k + yq - zp$ (completing the k-square).

$$D = \ell^2 - \Delta + i\epsilon, \quad \Delta = -xyq^2 + (1-z)^2 m^2 = \text{"effective mass term"}.$$

Have used: $x+y+z=1$

$$p^2 = m^2 = p'^2 = (p+q)^2 \Rightarrow 2pq + q^2 = 0$$

Result: $\bar{u}(p') \underbrace{\delta T^{\mu\nu}(p', p)}_{\gamma^\mu - \gamma^\nu} u(p) = 2ie^2 \int \frac{d^4 \ell}{(2\pi)^4} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{D^3}$

$$\cdot \bar{u}(p') \left[\gamma^\mu \underbrace{\left(-\frac{\ell^2}{2} + (1-x)(1-y)q^2 + (1-4z+z^2)m^2 \right)}_{\text{leads to } F_1(q^2) \text{ when 1st line is included}} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \underbrace{(2m^2 z(1-z))}_{\text{leads to } F_2(q^2)} \right] u(p)$$

$$\int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta + i\epsilon)^3}$$

Wick rotation:

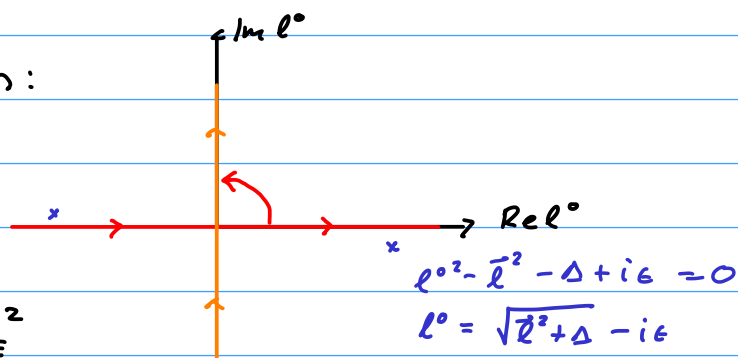
Then define ℓ_E : $\ell^0 = i\ell_E^0$

$$\vec{\ell} = \vec{\ell}_E$$

$$\Rightarrow \ell^2 = \ell^{02} - \vec{\ell}^2 = -\ell_E^{02} - \vec{\ell}_E^2 = -\ell_E^2$$

$$d^4 \ell = i d^4 \ell_E$$

$\uparrow$ with Euclidean metric!



$$\int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta + i\epsilon)^3} = i \int \frac{d^4 \ell_E}{(2\pi)^4} \frac{1}{(-\ell_E^2 - \Delta + i\epsilon)^3} = \frac{-i}{(2\pi)^4} \int_0^\infty d\ell_E \underbrace{2\pi^2 \ell_E^3}_{\text{surface 4d-sphere}} \frac{1}{(\ell_E^2 + \Delta - i\epsilon)^3}$$

$$= \frac{-i}{(4\pi)^2} \frac{1}{2\Delta}$$

$\uparrow$ Wolfram alpha

$$\underline{F_2(q^2)}$$

$$F_2(q^2) = 2ie^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{D^3} [2m^2 z(1-z)]$$

$$= \frac{\alpha}{2\pi} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2m^2 z(1-z)}{(1-z)^2 m^2 - xyq^2} + O(\alpha^2)$$

these diagrams have not been evaluated.

↑ $\frac{\alpha}{2\pi} = \frac{e^2}{4\pi 2\pi} = 2 \frac{e^2}{(4\pi)^2}$, $\int \frac{d^4l}{(2\pi)^4} \frac{1}{D^3} = \frac{-i}{(4\pi)^2} \frac{1}{2\Delta}$, $\Delta = (1-z)^2 m^2 - xyq^2$

$$F_2(0) = \frac{\alpha}{\pi} \int_0^1 dz dy dx \delta(x+y+z-1) \frac{z}{1-z} = \frac{\alpha}{\pi} \int_0^1 dz \int_0^{1-z} dy \frac{z}{1-z}$$

$$= \frac{\alpha}{\pi} \int_0^1 dz z = \frac{\alpha}{2\pi}$$

$\delta(x+y+z-1)$ contributes to x-integral

provided: $0 \leq 1-y-z \leq 1$

$$\Rightarrow -1 \leq -y-z \leq 0$$

$$\Rightarrow 0 \leq y+z \leq 1$$

We have

$$g = 2 + 2F_2(0) = 2 + \frac{\alpha}{\pi}$$

$$a_e = \frac{g-2}{2} = \frac{\alpha}{2\pi} \approx 0.001161 \quad (\text{Experiment: } a_e = 0.001160.)$$

Today: Best theoretical and experimental a_e agree to 8 significant figures!

$$\underline{F_1(q)}$$

First problem: ℓ -integral diverges: $\int d^4\ell \frac{\ell^2}{\ell^3} \sim \int d\ell \ell^3 \frac{\ell^2}{\ell^6} = \infty$.
UV-divergence.

Solution: Regularization and renormalization.

cutoff Λ in $\int d^4\ell$ a closer look shows that we are already
[PS: smooth cutoff, done the job; field strength renormalization.
Pauli-Villards
regularization.]

(Confession: We have omitted the factor $\sqrt{Z}^{m+n}$ which we obtained when we simplified the LSZ-formula with amputation ($m_0 \rightarrow m$). So far this has been ok since $Z = 1 + O(\alpha)$. Also ok for $F_2(q^2)$ since $F_2(q^2) = O(\alpha)$.

We have $T^\mu(p', p) = \text{diagram}$, but we need

$$\sqrt{Z}^2 T^\mu = Z T^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q^\nu}{2m} F_2(q^2) \quad (\text{redf. of } F_1 \text{ and } F_2)$$

$$Z T^\mu = (1 + \delta Z)(\gamma^\mu + \delta T^\mu) = \gamma^\mu + \delta T^\mu + \gamma^\mu \delta Z$$

$\uparrow$ correction by ignoring Z
 $\uparrow$ new term

$$F_1(q^2) = 1 + \delta F_1(q^2) + \delta Z = 1 + \delta F_1(q^2) - \delta F_1(0) = -\delta F_1(0) \quad (\text{PS 7.1}).$$

Infrared divergence

Second problem: IR divergence. Regularize by $\frac{-ig_{\mu\nu}}{q^2 + i\epsilon} \rightarrow \frac{-ig_{\mu\nu}}{q^2 - \mu^2 + i\epsilon}$.

$$\dots \Rightarrow F_1(q^2) = 1 + \frac{\alpha}{2\pi} \int_0^1 dx dy dz \delta(x+y+z-1) \cdot \left[\log \left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2 xy} \right) + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2 xy + \mu^2 z} \right] \Big|_{\mu \rightarrow 0} \xrightarrow{\mu \rightarrow 0} \infty \text{ for } z=1, x=y=0$$
$$- \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \mu^2 z} \Big|_{\mu \rightarrow 0} \xrightarrow{\mu \rightarrow 0} \infty + O(\alpha^2)$$

Want to evaluate the divergence. Ignore bounded terms.

$x=y=0$ and $z=1$ in numerator.

Perform the integration ... $\Rightarrow F_1(q^2) = 1 - \frac{\alpha}{2\pi} \underbrace{f_{IR}(q^2)} \log \left(\frac{-q^2}{\mu^2} \right) + O(\alpha^2)$

$$f_{IR}(q^2) = \int_0^1 \frac{m^2 - \frac{q^2}{2}}{m^2 - q^2 \xi(1-\xi)} d\xi - 1$$

$\rightarrow \log \left(\frac{-q^2}{\mu^2} \right)$ as $-q^2 \rightarrow \infty$.

Simple evaluation

Since $T^{\mu\nu} = i^{\mu} F_1(q^2) + \frac{i\sigma^{\mu\nu} q^{\nu}}{2m} F_2(q^2)$, we get an IR divergence term in iM .

$$\Rightarrow \frac{d\sigma(p \rightarrow p')}{d\Omega} = \left(\frac{d\sigma}{d\Omega} \right)_0 \left[1 - \frac{\alpha}{\pi} \log \left(\frac{-q^2}{m^2} \right) \log \left(\frac{-q^2}{\mu^2} \right) + O(\alpha^2) \right], \quad -q^2 \text{ large}$$

$$\frac{d\sigma(p \rightarrow p' + t)}{d\Omega} = \left(\frac{d\sigma}{d\Omega} \right)_0 \left[+ \frac{\alpha}{\pi} \log \left(\frac{-q^2}{m^2} \right) \log \left(\frac{-q^2}{\mu^2} \right) + O(\alpha^2) \right], \quad - " -$$

Sum is finite, and not dependent on μ !

The electron self-energy ($m_0 \rightarrow m$)

In ϕ^4 -theory we have found

$$\int d^4x e^{ipx} \langle \Omega | T \varphi(x) \varphi(0) | \Omega \rangle = \sum_{\lambda} \frac{i z_{\lambda}}{p^2 - m_{\lambda}^2 + i\epsilon} \quad (*)$$

Sum over internal degrees of freedom of the state,
if it is a single-part. state, two-part. state etc.

and

$$\int d^4x e^{ipx} \langle \Omega | T \phi(x) \phi(0) | \Omega \rangle = \text{sum of all diagrams using Feynman rules in mom. space, but without } e^{ipx} \text{ and } e^{-ipy} \text{ factors of external points, and without } p\text{-integration.}$$

Thus, for $\vec{p}^2 \sim m^2$ (= single part. mass) we have

$$\begin{aligned} \Sigma \text{ all diagrams, two lgs} &= \text{---} + \text{---} \circ \text{---} + \text{---} \bigcirc \text{---} + \dots \\ &= \frac{i}{p^2 - m_0^2 + i\epsilon} + \dots + \dots + \dots \\ &= \frac{i\epsilon}{p^2 - m^2 + i\epsilon} \end{aligned}$$

QED: Consider $\int d^4x e^{ipx} \langle \Omega | T \psi(x) \bar{\psi}(0) | \Omega \rangle :$

$$\begin{array}{c}
 \text{---} \leftarrow \text{---} \quad + \quad \text{---} \leftarrow \text{---} \leftarrow \text{---} \leftarrow \text{---} \quad + \dots \\
 p \qquad \qquad \qquad p \qquad k \qquad p \\
 \\
 \frac{i(p+m_0)}{p^2 - m_0^2 + i\epsilon} \quad + \quad \frac{i(p+m_0)}{p^2 - m_0^2} [-i\Sigma_2(p)] \frac{i(p+m_0)}{p^2 - m_0^2} \quad + \dots
 \end{array}$$

free-field prop. electron self-energy

where $-i\Sigma_2(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(k+m_0)}{k^2 - m_0^2 + i\epsilon} \gamma_\mu \frac{-i}{(p-k)^2 - \mu^2 + i\epsilon}$ (2nd order Σ_2
any order Σ)

$\Sigma_2(p)$ is regarded as a function of p ,
since $p^2 = p^2$.

↑ regularize apparent IR-div.,
but we can let $\mu \rightarrow 0$
(no IR-div.)

- introduce Feynman-parameters
- complete square, shifted momentum $\ell = k - xp$.
- symmetry num. integrals
- Pauli-Villars cutoff Λ mom. int.
- Wick-rotate

$$\text{Result: } \Sigma_2(p) = \frac{\kappa}{2\pi} \int_0^1 dx (2m_0 - xp) \log \left(\frac{x\Lambda^2}{(1-x)m_0^2 + xp^2 - x(1-x)p^2} \right)$$

Define one-particle irreducible diagrams (1PI): Any diagram that cannot be split in two by removing a single line:

 is 1PI, but not .

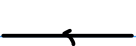
Let $-i\Sigma(p)$ be the sum of all 1PI diagrams:

$$-i\Sigma(p) = \text{-(1PI)} = \text{} + \text{} + \text{} + \dots$$

$-i\Sigma_2(p)$

(Should not be with external lines, since the are included explicitly below.)

$$\int d^4x e^{ipx} \langle \Omega | T \psi(x) \bar{\psi}(0) | \Omega \rangle = \text{}$$

$$= \text{} + \text{} + \text{} + \dots$$

$$= \frac{i(p + m_0)}{p^2 - m_0^2} + \frac{i(p + m_0)}{p^2 - m_0^2} [-i\Sigma(p)] \frac{i(p + m_0)}{p^2 - m_0^2} + \dots$$

$$= \frac{i}{p - m_0} \left(1 + \frac{\Sigma(p)}{p - m_0} + \dots \right) = \frac{i}{p - m_0} \frac{1}{1 - \frac{\Sigma(p)}{p - m_0}} = \frac{i}{p - m_0 - \Sigma(p)} \quad (\text{4.11})$$

Used that Σ and p commute

Sloppy notation, but easy to do properly.

Convergence of geometric series: Fix Λ , and let α be sufficiently small.

After the sum has been done, α can be increased (analytic continuation).

Pole is shifted! Pole when $(p^2 - m^2)(p+m)^{-1} = p-m=0$.

From $p=m_0$ to $p=m = m_0 + \Sigma(p=m)$

For QED (*) becomes:

$$\begin{aligned} \int d^4x e^{ipx} \langle \Omega | T \psi(x) \bar{\psi}(0) | \Omega \rangle &= \frac{i Z(p+m)}{p^2 - m^2 + i\epsilon} + \dots \\ &= \frac{i Z}{p-m + i\epsilon} = \frac{i}{Z(p) \quad p - m_0 - \Sigma(p)} \end{aligned}$$

(multiparticle terms)

Close to the pole $p=m$: Denom. = $(p-m) (1 - \Sigma'(p))_{p=m}$

$$\Rightarrow \underline{Z^{-1} = 1 - \Sigma'(p=m)}$$

To order $O(\alpha)$:

$$\begin{aligned} \delta m = m - m_0 &= \Sigma_2(p=m) \approx \Sigma_2(p=m_0) = \frac{\alpha m_0}{2\pi} \int_0^1 dx (2-x) \log \frac{x \Lambda^2}{(1-x)^2 m_0^2} \\ &= \frac{\alpha m_0}{2\pi} \int_0^1 (2-x) \left[\log \frac{x}{(1-x)^2} + \log \frac{\Lambda^2}{m_0^2} \right] dx \\ &= \underbrace{\frac{3\alpha}{2\pi}}_{\sim \frac{1}{300}} m_0 \left[\frac{1}{4} + \log \frac{\Lambda}{m_0} \right], \quad \text{UV divergent...} \end{aligned}$$

Logarithmic divergences are weak: $\frac{1}{300} \log \frac{\Lambda}{m_0} = 1 \Rightarrow \frac{\Lambda}{m_0} = e^{300} \approx 10^{130}!$

Note!! After renormalization:

m is the fixed, physical mass (independent of Λ).

m_0 is dependent on Λ .

For reasonable values of Λ , $\frac{\delta m}{m} \sim \frac{\delta m}{m_0}$ is small.

Renormalization of electric charge: overview and result

Consider corrections to the photon propagator $\sim$.

Vacuum polarization diagram

$$\begin{array}{c} \text{Diagram: a photon line with a fermion loop} \\ \sim = (-ie)^2 (-1) \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[\gamma^\mu \frac{i}{k-m} \gamma^\nu \frac{i}{k+q-m} \right] \equiv i\Pi_2^{\mu\nu}(q) \end{array}$$

$\uparrow$ fermion loop

In general:

$$\sim \text{(1PI)} \sim \equiv i\Pi^{\mu\nu}(q) = \sum \text{all 1PI into the photon prop.}$$

$$\Pi^{\mu\nu}(q) = C q^\mu q^\nu + D g^{\mu\nu}. \quad \text{Ward's identity: } q_\mu \Pi^{\mu\nu} = 0$$

$$C q^2 q^\nu + D q^\nu = 0$$

$$D = -C q^2$$

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \Pi(q^2)$$

Similarly to the electron self-energy:

$$\begin{aligned} \sim \text{(Dressed)} \sim &= \sim + \sim \text{(1PI)} \sim + \sim \text{(1PI)} \sim \text{(1PI)} \sim + \dots \\ &= \frac{\sim}{1 - \text{(1PI)} \sim} = \dots = \frac{-ig_{\mu\nu}}{q^2(1 - \Pi(q^2))}. \end{aligned}$$

(essentially; for S-matrix calculations)

Low q^2 -process ($q^2 \ll m^2$):

$$\sim \text{(Dressed)} \sim \sim \frac{Z_3(-ig_{\mu\nu})}{q^2}, \quad Z_3 = \frac{1}{1 - \Pi(0)}.$$

$\Pi(0)$ is dep. on reg. parameter, and $\Pi(0) \rightarrow \infty$ as r.p. $\rightarrow$ original theory

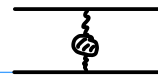
Note that even though $q_0^2 \ll m^2$, the virtual electron-positron pair has important and measurable consequences!

"photon energy"

Application / example:



interaction
between
two electrons,
lowest order



same, but with
vacuum pol. to
all orders

$$\frac{e^2 (-ig_{\mu\nu})}{q^2}$$



$$\frac{e^2 Z_3 (-ig_{\mu\nu})}{q^2} \equiv \frac{e^2 (-ig_{\mu\nu})}{q^2}$$

Renormalization:

$$e_0$$



$$e_0 \sqrt{Z_3} = e$$

If $q^2 \ll m^2$:

$$e_0$$



$$\frac{e_0}{\sqrt{1-\Pi(q^2)}} = e \frac{\sqrt{1-\Pi(0)}}{\sqrt{1-\Pi(q^2)}}$$

Thus the apparent charge is dependant on q^2 .

$\Rightarrow$ Effective potential from an electron is

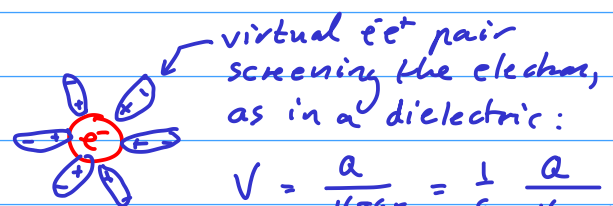
$$V(r) = -\frac{\alpha}{r} \left(1 + \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2mr}}{(mr)^{3/2}} + \dots \right), \quad r \gtrsim \frac{1}{m} \quad (= \text{Compton wavelength } 2.4 \mu\text{m, v.s. } \sim 100 \mu\text{m size of H})$$

At $r \ll 1/m$: One starts to see the bare charge e_0

At $r \gg 1/m$: One sees e ($\ll e_0$)

Observable as a shift in the energy levels of H.

(Part of the Lamb shift.)



$$V = \frac{a}{4\pi\epsilon r} = \frac{1}{\epsilon_r} \frac{a}{4\pi\epsilon_0 r},$$

V reduced by a factor of $\frac{1}{\epsilon_r}$.

