

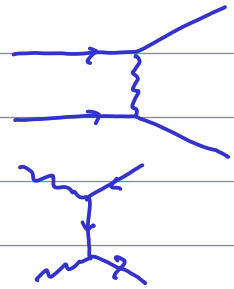
# The scattering (S) matrix

Interaction may lead to

- scattering (e.g., electron-electron-scattering)
- particle generation
- self-interaction



QED



Most introductory textbooks pretend that the particles are free if they are well separated.

Lecture notes overlapping partially with: PS 102-104 (4.71).

4.6 except p. 115.

7.2 p. 222-226.

## Eigenstates of interacting theories (scalar)

Hamiltonian  $H = P^0$ ,  $\vec{P}$ . Poincaré symmetry (problem set 4, pr. 3).

Assumptions: 1. Unique, translation- and Lorentz-inv. vacuum state  $|\Omega\rangle$ .

Energy zero chosen such that  $H|\Omega\rangle = 0$ .

2.  $\langle\Omega|\phi(x)|\Omega\rangle = 0$ . (problem set 8)

Poincaré symmetry:  $[H, \vec{P}] = 0$ , so there are  $|\lambda_{\vec{p}}\rangle$  with

$$H|\lambda_{\vec{p}}\rangle = E_{\vec{p}}^{\lambda} |\lambda_{\vec{p}}\rangle$$

$$\vec{P}|\lambda_{\vec{p}}\rangle = \vec{p} |\lambda_{\vec{p}}\rangle$$

Let  $m_{\lambda}$  be the rest energy (mass) of the states.

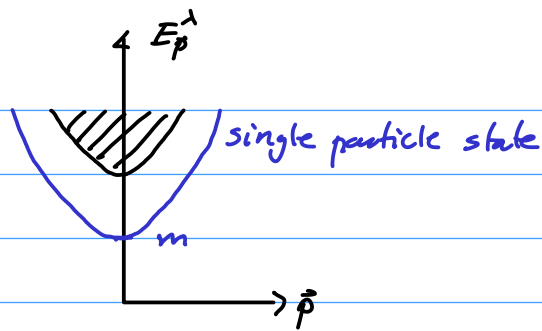
$$(m_{\lambda}, 0) \xrightarrow{\Lambda} p = (p^0, \vec{p}), \quad p^0{}^2 = \vec{p}^2 + m_{\lambda}^2$$

rest frame                      another frame

$$U^{\dagger} P^{\mu} U = \Lambda^{\mu}_{\nu} P^{\nu} \quad (\text{problem set 4, pr. 3})$$

$$\begin{aligned} P^{\mu} U |\lambda_0\rangle &= U U^{\dagger} P^{\mu} U |\lambda_0\rangle = U \Lambda^{\mu}_{\nu} P^{\nu} |\lambda_0\rangle \\ &= \Lambda^{\mu}_0 m_{\lambda} U |\lambda_0\rangle \equiv p^{\mu} U |\lambda_0\rangle, \end{aligned}$$

so  $|\lambda_{\vec{p}}\rangle = U |\lambda_0\rangle$ , with 4-mom.  $(E_{\vec{p}}^{\lambda}, \vec{p})$ ,  $E_{\vec{p}}^{\lambda 2} = \vec{p}^2 + m_{\lambda}^2$



Translation:  $\varphi(x) = e^{iP_x} \varphi(0) e^{-iP_x}$ , so

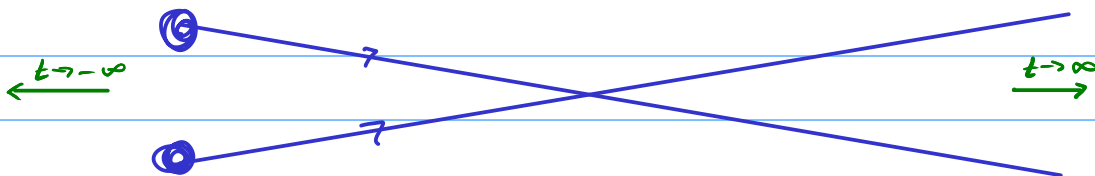
$$\begin{aligned} \langle \Omega | \varphi(x) | \lambda_{\vec{p}} \rangle &= \langle \Omega | e^{iP_x} \varphi(0) e^{-iP_x} | \lambda_{\vec{p}} \rangle \\ &= \langle \Omega | \varphi(0) | \lambda_{\vec{p}} \rangle e^{-iP_x} = \langle \Omega | U^\dagger \varphi(0) U | \lambda_0 \rangle e^{-iP_x} \\ &= \langle \Omega | \varphi(0) | \lambda_0 \rangle e^{-iP_x}, \quad p^0 = E_{\vec{p}}^\lambda = \sqrt{\vec{p}^2 + m_\lambda^2} \end{aligned}$$

When  $\lambda$  denotes a single particle  $|\lambda_{\vec{p}}\rangle$  is called  $|\vec{p}\rangle$ .

Def.  $Z = \langle \Omega | \varphi(0) | \vec{p}=0 \rangle^2$ , assumed real (by redefining  $|\lambda_0\rangle$ ).

[For free field  $\varphi(0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p + a_p^\dagger)$  and  $|\vec{p}\rangle = \sqrt{2E_p} a_p^\dagger |0\rangle$   
so  $Z = \langle 0 | \varphi(0) | \vec{p} \rangle^2 = 1$ .]

## Defining in- and out-states



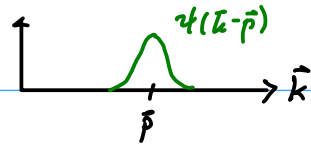
Interaction: When? Answer: Always.

Still we want the different particles not to interact with each other for the in-state (or out-state).

Must be localized (wave packets):

$$|\lambda_{\vec{p}}\rangle = \int \frac{d^3k}{(2\pi)^3} \psi(\vec{k}-\vec{p}) |\lambda_{\vec{k}}\rangle$$

$$|\vec{p}\rangle = \int \frac{d^3k}{(2\pi)^3} \psi(\vec{k}-\vec{p}) |\vec{k}\rangle.$$



Define operators:  $a_{\vec{p}}^{\dagger}(t) = i \int \Psi_{\vec{p}}^*(x) \vec{\partial}_0 \varphi(x)$   
 $a_{\vec{p}}^{\dagger\dagger}(t) = -i \int \Psi_{\vec{p}}(x) \vec{\partial}_0 \varphi(x)$

*[In general complicated preparations, but when  $t \rightarrow \pm\infty$  turns out to be similar to ladder ops.]*

with:  $\Psi_{\vec{p}}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{\psi(\vec{k}-\vec{p})}{\sqrt{2E_{\vec{k}}}} e^{-ikx}$ ,  $k^0 = E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$

*mass single part. interact. theory*

Free theory:  $a_{\vec{p}}^{\dagger}(t)$ ,  $a_{\vec{p}}^{\dagger\dagger}(t)$  evaluates to the usual wavepacket ladder op.:  $a_{\vec{p}}^{\dagger\dagger} = \int \frac{d^3k}{(2\pi)^3} \psi(\vec{k}-\vec{p}) a_{\vec{k}}^{\dagger}$   
 (exercise; see probl. set 9, pr. 12)

For  $t \rightarrow \pm\infty$  the  $\Psi_{\vec{p}}(x)$  and  $\Psi_{\vec{q}}(x)$  are spatially separated, and  $[a_{\vec{p}}^{\dagger}(t), a_{\vec{q}}^{\dagger\dagger}(t)] = 0$ ,  $\vec{p} \neq \vec{q}$ ,  $t \rightarrow \pm\infty$   
 (involves the field at different places in 3D-space, and  $[\varphi(\vec{x}), \varphi(\vec{y})] = 0$ ).

Limits: 1)  $t \rightarrow \pm\infty$

2)  $\psi(\vec{k}) \rightarrow 2\pi\delta(\vec{k})$  (plane wave limit)

Result:  $a_{\vec{p}}(\pm\infty)$ ,  $a_{\vec{p}}^{\dagger}(\pm\infty)$  work as ladder operators for vacuum and single part. (first step of ladder)

$$|\vec{p}\rangle = \frac{\sqrt{2E_{\vec{p}}}}{\sqrt{Z}} a_{\vec{p}}^{\dagger}(\pm\infty) |\Omega\rangle,$$

$$\sqrt{Z} = \langle \Omega | \varphi(0) | \vec{p}=0 \rangle$$

(derivation in lecture notes)

$$\langle \vec{p} | \vec{q} \rangle = 2E_{\vec{p}} (2\pi)^3 \delta(\vec{p}-\vec{q})$$

$$a_{\vec{p}}(\pm\infty) |\Omega\rangle = 0$$

Asymptotic in- and out-states:

$$|\vec{k}_1, \vec{k}_2, \dots\rangle_{\text{in}} = \prod_i \sqrt{\frac{2E_{\vec{k}_i}}{Z}} a_{\vec{k}_i}^{\dagger}(-\infty) |\Omega\rangle$$

$$|\vec{p}_1, \vec{p}_2, \dots\rangle_{\text{out}} = \prod_j \sqrt{\frac{2E_{\vec{p}_j}}{Z}} a_{\vec{p}_j}^{\dagger}(\infty) |\Omega\rangle$$

Note: The time dep. in  $a_p^\dagger(t)$  is not of the Heisenberg type.  
(No time dep. in the special case of free fields.)

Thus the  $t$  in  $a_p^\dagger(t)$  does not describe propagation.

$a_p^\dagger(-\infty)|\Omega\rangle$  is a state  $\sim$  free particle for  $t = -\infty$ .

$a_p^\dagger(\infty)|\Omega\rangle$  is a state  $\sim$  free particle for  $t = \infty$ , but messy at  $t = -\infty$ .

Scattering in the Schrödinger picture:

Let ref. time be  $t = -T$ .

Evolution from  $t = -T$  to  $t = +T$ :  $S = e^{-iH_2T}$ .

For  $t = -T$  in S.p.:  $|\vec{k}_1, \vec{k}_2, \dots\rangle = |\vec{k}_1, \vec{k}_2, \dots\rangle_{in} \leftarrow \text{Heisenberg p.}$   
 $|\vec{p}_1, \vec{p}_2, \dots\rangle_{out}$

For  $t = +T$  in S.p.:  $|\vec{p}_1, \vec{p}_2, \dots\rangle = S |\vec{p}_1, \vec{p}_2, \dots\rangle_{out}$

${}_{out}\langle \vec{p}_1, \vec{p}_2, \dots | \vec{k}_1, \vec{k}_2, \dots \rangle_{in} = {}_{out}\langle \vec{p}_1, \vec{p}_2, \dots | S^\dagger S | \vec{k}_1, \vec{k}_2, \dots \rangle_{in}$

$= \underbrace{\langle \vec{p}_1, \vec{p}_2, \dots |}_{\text{at } t=+T \text{ in S.p.}} S \underbrace{|\vec{k}_1, \vec{k}_2, \dots\rangle}_{\text{at } t=-T \text{ in S.p.}} : S\text{-matrix elements.}$

# LSZ reduction (Lehmann-Symanzik-Zimmermann)

Use wavepacket in- and out-states:

$$|k_1, k_2\rangle_{in} \equiv |in\rangle = A a_{k_1}^{\dagger}(-\infty) a_{k_2}^{\dagger}(-\infty) |\Omega\rangle, \quad A = \prod_i \sqrt{\frac{2E_{k_i}}{Z}}$$

$$|\bar{p}_1, \bar{p}_2, \dots\rangle_{out} \equiv |out\rangle = \prod_j \sqrt{\frac{2E_{\bar{p}_j}}{Z}} a_{\bar{p}_1}^{\dagger}(\infty) a_{\bar{p}_2}^{\dagger}(\infty) \dots |\Omega\rangle$$

S-matrix elements:

$$\begin{aligned} \langle out | in \rangle &= A \langle out | a_{k_1}^{\dagger}(-\infty) a_{k_2}^{\dagger}(-\infty) |\Omega\rangle \\ &= A \langle out | [a_{k_1}^{\dagger}(-\infty) - a_{k_1}^{\dagger}(\infty)] a_{k_2}^{\dagger}(-\infty) |\Omega\rangle \\ &\quad \text{[assuming } k_i \neq \bar{p}_j \text{ for all } j \text{ (also we must assume } k_i \neq \bar{p}_i \text{).}] \\ &= iA \left[ \lim_{t \rightarrow \infty} - \lim_{t \rightarrow -\infty} \right] \int d^3x_i \bar{\Psi}_{k_i}(x_i) \overleftrightarrow{\partial}_0 \langle out | \phi(x_i) a_{k_2}^{\dagger}(-\infty) |\Omega\rangle \end{aligned}$$

We have  $\left[ \lim_{t \rightarrow \infty} - \lim_{t \rightarrow -\infty} \right] \int d^3x f(x) \overleftrightarrow{\partial}_0 g(x) = \int d^4x \partial_0 (f(x) \overleftrightarrow{\partial}_0 g(x))$

$$\begin{aligned} &= \int d^4x [f(x) \partial_0^2 g(x) - (\partial_0^2 f(x)) g(x)] \\ &= \int d^4x [f(x) \partial_0^2 g(x) + [(m^2 - \nabla^2) f(x)] g(x)] \quad (\text{Klein-Gordon for } f(x)) \\ &= \int d^4x [f(x) (\Box + m^2) g(x)] \quad (\text{Int. by part twice.}) \end{aligned}$$

Assume  $f(x) \rightarrow 0$  suff. quickly as  $\vec{x} \rightarrow \infty$ , so surface terms vanish.)

$$= iA \int d^4x \bar{\Psi}_i(x_i) (\Box_{x_i} + m^2) \langle out | \phi(x_i) a_{k_2}^{\dagger}(-\infty) |\Omega\rangle \quad (*)$$

First step done. Effectively replaced  $a_{k_i}^{\dagger}$  by  $\phi(x_i)$ .

Continue the procedure (next step in lecture notes).

Result:

$$\begin{aligned} \langle out | in \rangle &= \prod_i \frac{i\sqrt{2E_{k_i}}}{\sqrt{Z}} \int d^4x_i \bar{\Psi}_{k_i}(x_i) (\Box_{x_i} + m^2) \\ &\quad \cdot \prod_j \frac{i\sqrt{2E_{\bar{p}_j}}}{\sqrt{Z}} \int d^4y_j \bar{\Psi}_{\bar{p}_j}^*(y_j) (\Box_{y_j} + m^2) \end{aligned}$$

$$\cdot \langle \Omega | T \phi(x_1) \phi(x_2) \dots \phi(y_1) \phi(y_2) \dots | \Omega \rangle$$

Only new feature in next step is  $T$ . Arises as follows:

$$\begin{aligned} &\langle out | \phi(x_1) a_{k_2}^{\dagger}(-\infty) |\Omega\rangle \\ &= \langle out | \phi(x_1) a_{k_2}^{\dagger}(-\infty) |\Omega\rangle \\ &\quad - \langle out | a_{k_2}^{\dagger}(\infty) \phi(x_1) |\Omega\rangle \\ &= \langle out | T \{ \phi(x_1) a_{k_2}^{\dagger}(-\infty) - \phi(x_1) a_{k_2}^{\dagger}(\infty) \} | \Omega \rangle \end{aligned}$$

Common error in textbooks:

$$\begin{aligned} A(\tau)B(0) &= T A(\tau)B(0) \\ &= T \{ A(\tau) + \int_0^\tau dt \dot{A}(t) \} B(0) \\ &= B(0)A(0) + \int_0^\tau \dot{A}(t) B(0) dt - A(0)B(0) \end{aligned}$$

Take the plane-wave limit  $\gamma(k) \rightarrow (2\pi)^3 \delta(k) \Leftrightarrow \Psi_k(x) \rightarrow \frac{1}{\sqrt{2E_k}} e^{-ikx}$

$$\langle \text{out} | \text{in} \rangle = \prod_i \frac{i}{\sqrt{2}} \int d^4 x_i e^{-ik_i x_i} (\not{\partial}_{x_i} + m^2)$$

$$\cdot \prod_j \frac{i}{\sqrt{2}} \int d^4 y_j e^{ip_j y_j} (\not{\partial}_{y_j} + m^2)$$

$$\cdot \langle \Omega | T \varphi(x_1) \varphi(x_2) \dots \varphi(y_1) \varphi(y_2) \dots | \Omega \rangle$$

$$= \prod_i \frac{1}{i\sqrt{2}} (k_i^2 - m^2) \int d^4 x_i e^{-ik_i x_i}$$

external momenta, so the  $(\ ) \rightarrow 0$ .

$$g(x) \xrightarrow{\mathcal{F}} G(k)$$

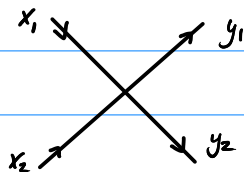
$$\square g(x) \leftrightarrow -k^2 G(k)$$

$$\cdot \prod_j \frac{1}{i\sqrt{2}} (p_j^2 - m^2) \int d^4 y_j e^{ip_j y_j}$$

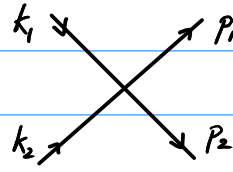
$$\cdot \langle \Omega | T \varphi(x_1) \varphi(x_2) \dots \varphi(y_1) \varphi(y_2) \dots | \Omega \rangle \leftarrow \sum \text{connected diagrams}$$

Fourier-transforming diagrams can be done by omitting exp.-factors and momentum-integrals of external points. Extra care is needed for those where the momentum is determined by the others.

Consider



$\xrightarrow{\mathcal{F}}$



$$\langle \Omega | \varphi(x_1) \varphi(x_2) \varphi(y_1) \varphi(y_2) | \Omega \rangle = (-i\lambda) \int \frac{d^4 k_1}{(2\pi)^4} e^{ik_1 x_1} \int \frac{d^4 k_2}{(2\pi)^4} e^{ik_2 x_2} \int \frac{d^4 p_1}{(2\pi)^4} e^{-ip_1 y_1}$$

$$\cdot \underbrace{e^{-i(k_1+k_2-p_1)y_2}}_{\int \frac{d^4 p_2}{(2\pi)^4} e^{-ip_2 y_2} (2\pi)^4 \delta(k_1+k_2-p_1-p_2)} D_F(k_1) D_F(k_2) D_F(p_1) D_F(k_1+k_2-p_1)$$

$\int \frac{d^4 p_2}{(2\pi)^4} e^{-ip_2 y_2} (2\pi)^4 \delta(k_1+k_2-p_1-p_2)$   $\leftarrow$  momentum space Feynman-propagator

Fourier-transforming:

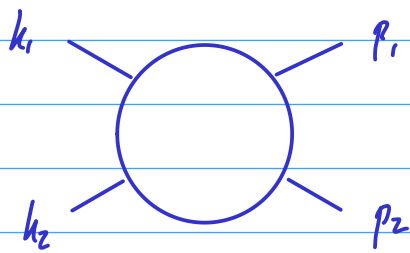
$$(-i\lambda) (2\pi)^4 \delta(k_1+k_2-p_1-p_2) D_F(k_1) D_F(k_2) D_F(p_1) D_F(k_1+k_2-p_1)$$

$$\langle p_1, p_2, \dots | S | k_1, k_2, \dots \rangle = (2\pi)^4 \delta(\sum k_i - \sum p_j) i\mathcal{M}(k_1, k_2, \dots, p_1, p_2, \dots) \quad (\text{again recall } k_i \neq p_j)$$

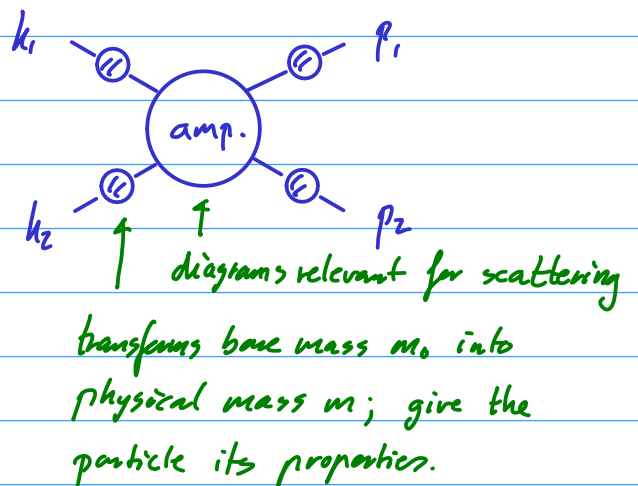
$\swarrow$  def.

$$i\mathcal{M} = \prod_i \frac{1}{i\sqrt{2}} (k_i^2 - m^2) \cdot \prod_j \frac{1}{i\sqrt{2}} (p_j^2 - m^2) \underbrace{\sum \text{connected diagrams}}$$

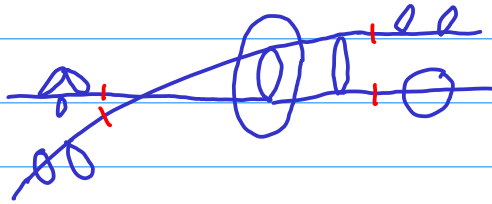
Feynman rules mom.-space, but not with exp's and mom. integrals for external points.



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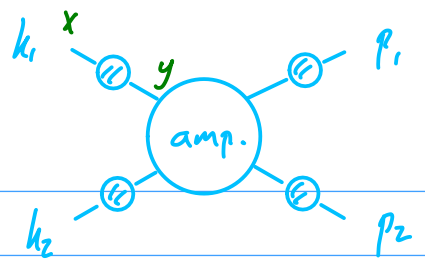
Def amputate: Start from the tip of an external line, find the last point where the diagram can be cut by removing a single prop. line. Cut there.



$$\begin{aligned} \sum(\text{connected diagrams, four legs}) &= \sum(\text{connected, amp. diag., four legs}) \\ &\cdot \sum(\text{connected diag. two legs}) \leftarrow \text{leg } k_1 \\ &\cdot \sum(\text{connected diag. two legs}) \leftarrow \text{leg } k_2 \\ &\cdot \dots \leftarrow \text{legs } p_1 \text{ and } p_2 \end{aligned}$$

Two-point corr. func.:

$$\int d^4x e^{ipx} \langle 0 | T \phi(x) \phi(0) | 0 \rangle = \frac{i}{p^2 - m^2 + i\epsilon} \quad (\text{free theory})$$



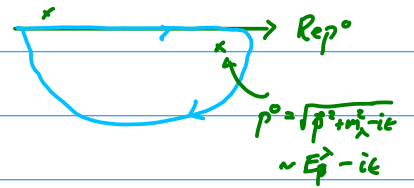
$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \langle \Omega | \phi(x) | \lambda_p \rangle \langle \lambda_p | \phi(y) | \Omega \rangle$$

$$\text{1 } 1 = 1\Omega|1\Omega + \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\lambda_p\rangle \langle \lambda_p|, \quad \langle \Omega | \phi(0) | \Omega \rangle = 0$$

$$= \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} Z_{\lambda}, \quad p^0 = E_p = \sqrt{m^2 + \vec{p}^2}, \quad Z_{\lambda} = |\langle \Omega | \phi(0) | \lambda_0 \rangle|^2$$

$$\text{2 } \langle \Omega | \phi(x) | \lambda_p \rangle = e^{-ipx} \sqrt{Z_{\lambda}}$$

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \sum_{\lambda} Z_{\lambda} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$



$$\int d^4x e^{ipx} \langle \Omega | T \phi(x) \phi(0) | \Omega \rangle = \sum_{\lambda} \frac{i Z_{\lambda}}{p^2 - m^2 + i\epsilon}$$

$$= \frac{i Z}{p^2 - m^2 + i\epsilon} + \text{terms finite at } p^2 = m^2$$

$$\sum(\text{connected diagrams, four legs}) = \sum(\text{connected, amp. diag., four legs})$$

$$\cdot \sum(\text{connected diagr. two legs})$$

$$\cdot \sum(\text{connected diagr. two legs})$$

...

$$= \sum(\text{connected, amp. diag., four legs})$$

$$\cdot \frac{iZ}{k_1^2 - m^2 + i\epsilon} \cdot \frac{iZ}{k_2^2 - m^2 + i\epsilon} \cdot \frac{iZ}{p_1^2 - m^2 + i\epsilon} \cdot \frac{iZ}{p_2^2 - m^2 + i\epsilon}$$

$$iM(k_1, k_2, p_1, p_2) = \left(\frac{1}{i\sqrt{Z}}\right)^4 (k_1^2 - m^2)(k_2^2 - m^2)(p_1^2 - m^2)(p_2^2 - m^2) \sum(\text{connected diagrams, four legs})$$

$$= \sqrt{Z}^4 \sum(\text{connected, amp. diag., four legs})$$

Feynman rules mom. space, but  
not with exp's and mom. integrals  
for external points.

## Alternative method of obtaining S-matrix Feynman rules

Nair guess (assuming  $k_i \neq p_j$ ):

$$\langle p_1 \dots p_n | S | k_A k_B \rangle = \langle p_1 \dots p_n | T \left( \exp[-i \int dt H_I(t)] \right) | k_A k_B \rangle_0 \quad \text{connected, amputated}$$

$e^{-i24t}$

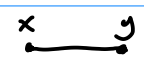
The external states complicate, since not only fully contracted terms of  $T \exp[\dots]$  contribute. We therefore construct artificial "fields"  $\varphi_A(-\infty) = \sqrt{2E_{k_A}}(a_{k_A} + a_{k_A}^\dagger)$ , ... which create the free in-state from free vacuum:  $\varphi_A \varphi_B |0\rangle = |k_A k_B\rangle_0$ :

$$\begin{aligned} \langle p_1 \dots p_n | S | k_A k_B \rangle &= \langle 0 | T \varphi_1(\infty) \varphi_2(\infty) e^{-i \int dt H_I(t)} \varphi_A(-\infty) \varphi_B(-\infty) | 0 \rangle \\ &= \text{fully contracted terms} \\ &\quad \uparrow \text{Wick} \end{aligned}$$

We write contractions with the artificial  $\varphi$ 's as

$$\begin{aligned} \overbrace{\varphi(x) | k_A \rangle_0} &\equiv \overbrace{\varphi(x) \varphi_A(-\infty)} = [\varphi(x), \varphi_A(-\infty)] \\ &\quad \uparrow x^0 > -\infty \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_p e^{-ipx}, \sqrt{2E_{k_A}} a_{k_A}^\dagger] = e^{-ik_A x} \quad \downarrow [a_p, a_{k_A}^\dagger] = (2\pi)^3 \delta(\vec{p} - \vec{k}_A) \\ \langle p_1 | \varphi(x) &= e^{ip_1 x} \end{aligned}$$

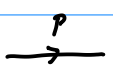
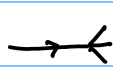
### Feynman - rules ( $\varphi^4$ ), pos. space

1.  =  $D_F(x-y)$
2.  =  $(-i\lambda) \int d^4 z$
3.  =  $e^{-ikx}$  (external line)
4. divide by 5.

Why not contractions between external states?

Answer: Such diagrams are not fully connected.

### mom. space

1.  =  $D_F(p)$
  2.  =  $-i\lambda$
  3.  = 1
  4. mom. cons. vertices
  5. integrate undetermined loop mom.  $\int \frac{d^4 p}{(2\pi)^4}$
  6. divide by 5
- same as those proved with LSE. Confirm our Nair guess.

3. Note that the  $e^{-ikx}$  is used (together with the other exp's from Fourier-repr. of propagators) via vertex integration to obtain mom. conserving  $\delta$ -function. ( $x$  is a vertex, not external point.)