

University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 2

Problem 1 Two scalar fields (T. Bringmann)

Consider two interacting, real, scalar fields ϕ_1 and ϕ_2 , described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) - \frac{1}{2}m_1^2 \phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2}m_2^2 \phi_2^2 - \lambda \phi_1^2 \phi_2^2, \quad (1)$$

where λ is a constant.

Derive the equations of motion for the theory (1). Do so by directly using the principle of least action (by varying the fields), i.e., not by using the Euler-Lagrange equations. Give an explanation of why the boundary terms do not contribute in your result, and discuss the relation to the Klein-Gordon-equation(s) you would expect for non-interacting fields.

Problem 2 Obtaining ladder operators from fields

(J. Skaar)

a) The hermitian Klein-Gordon field $\phi(x)$ and its canonical momentum density $\pi(x)$ are

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right), \quad (2a)$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{-ipx} - a_{\mathbf{p}}^\dagger e^{ipx} \right), \quad (2b)$$

where $p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. Find a method to invert these relations; thus obtaining the ladder operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ from $\phi(x)$ and $\pi(x)$.

b) By requiring equal-time commutators $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$, show that we must have

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}), \quad (3a)$$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0. \quad (3b)$$

c) Prove that

$$a_{\mathbf{p}} = i \int d^3x \Psi_{\mathbf{p}}^*(x) \overleftrightarrow{\partial}_0 \phi(x) \quad (4)$$

for $x^0 = 0$. Here we have defined

$$\Psi_{\mathbf{p}}(x) = \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{-ipx}, \quad p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \quad (5)$$

and the symbol $\overleftrightarrow{\partial}_0$, which operates “in both directions”:

$$f \overleftrightarrow{\partial}_0 g = f \partial_0 g - (\partial_0 f) g. \quad (6)$$

We will use expression (4) later in the course, when discussing the important LSZ formula.

Problem 3 The complex Klein-Gordon field (Peskin & Schroeder exercise 2.2, modified)

This problem considers the quantization of a *complex*, scalar field. It is both a good recap on how we quantized the real scalar field, and at the same time needs concepts that we meet when quantizing fermions (in particular the necessary appearance of anti-particles). The Lagrangian is

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi.$$

It is easiest to analyze this theory by considering ϕ and ϕ^* (rather than the real and imaginary parts) as the independent, dynamical variables.

a) Show that the Hamilton operator is given by

$$H = \int d^3x (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi)$$

b) Compared to the real case, the theory exhibits an additional symmetry, $\phi \mapsto e^{-i\alpha} \phi$. Show that this is indeed a symmetry, and use Noether's theorem to show that the corresponding conserved charge is proportional to

$$Q = -i \int d^3x (\dot{\phi}^* \phi - \phi^* \dot{\phi}). \quad (7)$$

So far we have considered the classical fields ϕ and ϕ^* . We now quantize the theory, by promoting the fields to operators ϕ and $\phi^\dagger$, respectively. Similarly, the canonical momentum densities are π and $\pi^\dagger$. Complex conjugation (*) in the classical expressions above must therefore be replaced by hermitian conjugate (†).

We require

$$\begin{aligned} [\phi(\mathbf{x}), \pi(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \\ [\phi^\dagger(\mathbf{x}), \pi^\dagger(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (8)$$

while

$$\begin{aligned} [\phi(\mathbf{x}), \phi(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \pi(\mathbf{y})], \\ [\phi(\mathbf{x}), \phi^\dagger(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \pi^\dagger(\mathbf{y})], \\ [\phi(\mathbf{x}), \pi^\dagger(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \phi^\dagger(\mathbf{y})]. \end{aligned} \quad (9)$$

c) In the Heisenberg picture the fields evolve according to the Heisenberg equations of motion,

$$i \frac{\partial \phi(x)}{\partial t} = [\phi(x), H] \quad (10a)$$

$$i \frac{\partial \pi(x)}{\partial t} = [\pi(x), H]. \quad (10b)$$

Prove that ϕ and π satisfy the following Hamilton's equations of motion:

$$\frac{\partial \phi(x)}{\partial t} = \pi^\dagger(x), \quad (11a)$$

$$\frac{\partial \pi(x)}{\partial t} = (\nabla^2 - m^2) \phi^\dagger(x). \quad (11b)$$

From these deduce that ϕ satisfies the Klein-Gordon equation.

d) Introduce annihilation and creation operators, and show that

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}) \quad (12)$$

if we ignore an infinite constant (as before). Here you should convince yourself that the Fourier decomposition of a complex function $\phi(\mathbf{x})$, already at the classical level, requires twice as many independent Fourier coefficients as for a real function. When promoted to operators, you should thus denote those with, e.g., $a_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$ (rather than $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$).

Argue that the theory contains two types of particles with mass m .

e) Rewrite the conserved charge (7) in terms of the annihilation and creation operators introduced above. How does the charge between the two types of particles differ?

Problem 4 Complex fields (optional; for the interested student)

(J. Skaar)

In the lectures, and above, we have treated ϕ and ϕ^* as independent. We will now justify this method, which is far from obvious, since ϕ^* in fact is uniquely determined by ϕ .

We write

$$\phi = \phi_r + i\phi_i, \quad (13)$$

where ϕ_r and ϕ_i are the real and imaginary parts of ϕ , respectively. Clearly, we may treat ϕ_r and ϕ_i as independent fields, that can be varied separately to obtain Euler-Lagrange equations for ϕ_r and ϕ_i .

Instead of ϕ_r and ϕ_i as independent “coordinates” in the complex ϕ -plane, let us fix ϕ_0 , and use real parameters s and t to express

$$\phi_r = \frac{1}{2}[s\phi_0 + t\phi_0^*], \quad (14a)$$

$$\phi_i = \frac{1}{2i}[s\phi_0 - t\phi_0^*], \quad (14b)$$

or

$$s\phi_0 = \phi_r + i\phi_i, \quad (15a)$$

$$t\phi_0^* = \phi_r - i\phi_i, \quad (15b)$$

This gives

$$\frac{1}{\phi_0} \frac{\partial}{\partial s} = \frac{\partial}{\partial(s\phi_0)} = \frac{1}{2} \left[\frac{\partial}{\partial\phi_r} - i \frac{\partial}{\partial\phi_i} \right], \quad (16a)$$

$$\frac{1}{\phi_0^*} \frac{\partial}{\partial t} = \frac{\partial}{\partial(t\phi_0^*)} = \frac{1}{2} \left[\frac{\partial}{\partial\phi_r} + i \frac{\partial}{\partial\phi_i} \right]. \quad (16b)$$

We therefore write

$$\frac{\partial}{\partial\phi} = \frac{1}{2} \left[\frac{\partial}{\partial\phi_r} - i \frac{\partial}{\partial\phi_i} \right], \quad (17a)$$

$$\frac{\partial}{\partial\phi^*} = \frac{1}{2} \left[\frac{\partial}{\partial\phi_r} + i \frac{\partial}{\partial\phi_i} \right], \quad (17b)$$

but have in mind that they actually mean (16). For example, $\frac{\partial}{\partial\phi}$ means that one varies $s\phi_0$ while holding $t\phi_0^*$ constant.

a) Prove that $\frac{\partial\phi}{\partial\phi} = \frac{\partial\phi^*}{\partial\phi^*} = 1$ and $\frac{\partial\phi}{\partial\phi^*} = \frac{\partial\phi^*}{\partial\phi} = 0$.

b) Argue that the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)} = 0, \quad (18a)$$

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0, \quad (18b)$$

imply

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0, \quad (19a)$$

$$\frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = 0. \quad (19b)$$

c) Go through the derivation of Noether's theorem, and convince yourself that the method of treating ϕ and ϕ^* as independent fields gives the same result as if you consider ϕ_r and ϕ_i as the independent fields.