University of Oslo

FYS4170/9170 – Relativistic Quantum Field Theory

Problem set 8

Problem 1 Wick's theorem and contractions (J. Skaar)

When computing correlation functions, it is useful to decompose the fields into positive and negative frequency parts. For the field in the interaction picture, this amounts to

$$\phi(x) = \phi^+(x) + \phi^-(x) \tag{1}$$

with

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}} e^{-ipx} \tag{2}$$

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a^\dagger_{\mathbf{p}} e^{ipx} \tag{3}$$

such that

$$\phi^+(x)|0\rangle = 0\,, \qquad \langle 0|\phi^-(x) = 0\,. \tag{4}$$

a) Prove Wick's theorem for two fields:

$$T\phi(x_1)\phi(x_2) = N\{\phi(x_1)\phi(x_2) + \overbracket{\phi(x_1)\phi}(x_2)\}, \tag{5}$$

where

$$\overbracket{\phi(x_1)\phi}(x_2) = \begin{cases} [\phi^+(x_1), \phi^-(x_2)], & \text{if } x_1^0 > x_2^0, \\ [\phi^+(x_2), \phi^-(x_1)], & \text{if } x_2^0 > x_1^0. \end{cases} \tag{6}$$

Also prove that

$$\overbracket{\phi(x_1)\phi}(x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle = D_F(x_1 - x_2). \tag{7}$$

b) Use Wick's theorem to obtain an expression for the vacuum expectation value

$$\langle 0|T\{\phi(x_1)\phi(x_2)\cdots\phi(x_m)\}|0\rangle. \tag{8}$$

What is the value of the vacuum expectation value if there is an odd number m of fields?

> **Hint:** With an odd number of fields, it is not possible to contract all fields. Then consider what happens after evaluating the vacuum expectation value.

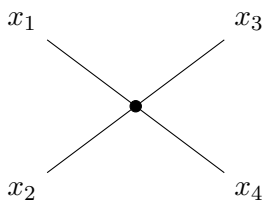
c) Calculate $\langle\Omega|\phi(x)|\Omega\rangle$ for ϕ^4-theory. Here $\phi(x)$ is the field in the Heisenberg picture (not interaction picture as above!).

Hint: See (4.31) in P&S, where we put $T = \infty$ according to the derivation in the lectures. First note that the derivation is valid for one field. Then consider the numerator, and count the number of fields in the terms after expanding the exponential.

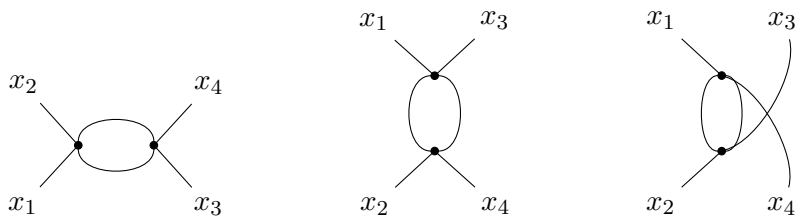
d) Go through the induction proof of Wick's theorem in P&S, p. 90. Be aware of the "paradox" $1 = N\{1\} = N\{aa^\dagger - a^\dagger a\} = a^\dagger a - a^\dagger a = 0$, which tells us that it is not allowed to insert the value of a commutator inside the $N\{\cdot\}$.

Problem 2 Feynman diagrams and symmetry factors (T. Klungland)

This problem explores some fully connected diagrams[1] contributing to the ϕ^4 4-point function, denoted by $\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|\Omega\rangle$. At the leading order, it is given by the diagram[2]



At order $\mathcal{O}(\lambda^2)$ there are three contributing diagrams:



We denote these by S, T, and U respectively.

a) Using the Feynman rules on page 94 in P&S, write down the position space representation of each of these diagrams. Do *not* attempt to perform the integrals (unless you have entirely too much spare time), simply leave them undetermined.

Solution: The symmetry factor for each of the diagrams is 2, since the two propagators connecting the two parts of the diagram can be interchanged without affecting anything.

[1] In a *fully connected* diagram all lines are connected to every external point by a sequence of lines and vertices. As you will see later in the course, these are the only diagrams that end up contributing to the correlation functions, as all disconnected diagrams factor out in both the numerator and denominator of Eq. (4.31) in P&S and thus cancel.

[2] In all diagrams in this exercise we place dots on the vertices in order to make it more explicit when two lines "accidentally" cross each other and when there is a genuine vertex.

Denoting the positions of the vertices by w and z, which are integrated over, the diagrams take the values

$$\begin{aligned}
S &= \frac{(-i\lambda)^2}{2}\int d^4z d^4w D_F(z-x_1)D_F(z-x_2)D_F^2(w-z) \\
&\quad \times D_F(x_3-w)D_F(x_4-w), \\
T &= \frac{(-i\lambda)^2}{2}\int d^4z d^4w D_F(z-x_1)D_F(w-x_2)D_F^2(w-z) \\
&\quad \times D_F(x_3-z)D_F(x_4-w), \\
U &= \frac{(-i\lambda)^2}{2}\int d^4z d^4w D_F(z-x_1)D_F(w-x_2)D_F^2(w-z) \\
&\quad \times D_F(x_3-w)D_F(x_4-z).
\end{aligned}$$

b) The symmetry factors of the diagrams in the previous part can be derived by Taylor expanding the numerator of Eq. (4.31) in P&S (we have four external fields instead of two; otherwise the equation can be used directly). At $\mathcal{O}(\lambda^2)$ we have, dropping the I subscripts,

$$\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2\int d^4z d^4w \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(z)\phi^4(w)\}|0\rangle,$$

where the factor $1/2!$ appears due to the Taylor expansion. The symmetry factor is found by counting the number of equivalent contractions corresponding to each particular diagram; when multiplying the above expression by this number, the symmetry factor s can be read off from the prefactor to the integral as $-\lambda^2/s$.

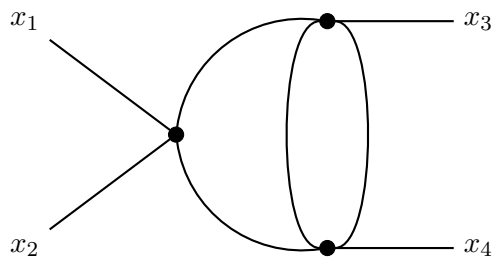
Use this to explicitly verify the symmetry factor of the S diagram (the process for the other two is identical).

Solution: The field at x_1 can be contracted with each of the four fields at z, i.e. 4 possible contractions; this leaves 3 possibilities to contract the field at x_2 with the ones at z. The same goes for contracting the fields at x_3 and x_4 with the ones at w. Finally, there are 2 ways to pair the remaining fields at z and w with one another. Now noting that since w and z are integrated over, we would get exactly the same results by swapping $w \leftrightarrow z$ in the above arguments; thus all of these contractions can effectively be "counted twice." Collecting the various combinatoric factors, the symmetry factor is given by

$$s = \frac{(4!)^2 2!}{4^2\cdot 3^2\cdot 2^2} = 2,$$

like before.

c) Repeat the above process for a sample $\mathcal{O}(\lambda^3)$ diagram, given by



i.e. find the symmetry factor of the diagram both directly by looking at the symmetries of the diagram, and from counting contractions.

Solution: Writing out the relevant term in the Taylor expansion like in the previous part, we have

$$\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \int d^4w d^4y d^4z \\ \times \langle 0|T\big\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(w)\phi^4(y)\phi^4(z)\big\}|0\rangle.$$

As in the previous problem, the three vertices can be interchanged, which will lead to 3! equivalent contractions; for one specific example, we denote the positions of the vertices by (clockwise, starting with the vertex connected to x_1 and x_2) w, y, and z.

Now, there are 4×3 ways of contracting fields at x_1 and x_2 into the vertex at w, and similarly 4×4 ways of contracting the fields at x_3 and x_4 into the vertices at y and z. Next, we can contract one of the remaining fields at w with one at y in 2×3 ways, and the last field at w with one at z in 3 ways. Lastly there are 2 ways to contract the fields at y and z with one another.

Putting all of this together, the symmetry factor is given by

$$s = \frac{(4!)^3 3!}{4^3 \cdot 3^3 \cdot 2^2 \cdot 3!} = 2.$$

This is the same as what we find by studying the diagram; the diagram is symmetric with regard to interchange of the two propagators between y and z (as they were named in the above discussion), giving a symmetry factor of 2 as there are no other symmetries.