University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 9

Problem 1 S matrix and LSZ reduction (J. Skaar)

a) For the free Klein-Gordon field, verify that the formula

$$a^{\psi}_{\mathbf{p}} = i \int d^3x \Psi^*_{\mathbf{p}}(x) \overleftrightarrow{\partial}_0 \phi(x), \quad \Psi_{\mathbf{p}}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{\psi(\mathbf{k}-\mathbf{p})}{\sqrt{2E_{\mathbf{k}}}} e^{-ikx}, \tag{1}$$

gives a time-independent wavepacket operator with spectrum $\psi(\mathbf{k}-\mathbf{p})$. Compare with the wavepacket ladder operator in problem set 3.

> **Hint:** See problem set 2 (where the same calculation is done for the single-momentum ladder operator).

b) Again for the free Klein-Gordon field, prove that $Z = 1$ (see the definition of Z in Sec. 1 of the lecture notes).

c) Take a look at Fig. 2 in the lecture note "The S matrix and the LSZ reduction formula". The figure indicates the wave function associated with the in-state, for a large negative t. Draw similarly the wave function associated with the out-state, for a large positive t. For the case with negligible interaction, also draw the wave function of the in-state, for a large positive t, and the wave function of the out state, for a large negative t. When there is interaction, which of the four pictures (in- and out-state wave functions combined with large positive or negative t) do you think will be "messy"?

> **Hint:** To verify that your sketches are correct, note that the two for a large positive t should be equal when there is negligible interaction, since the S matrix element then is 1. Also the two for a large negative t should be equal in this case.

> **Solution:**
> The in-state wavefunction for large, negative t:
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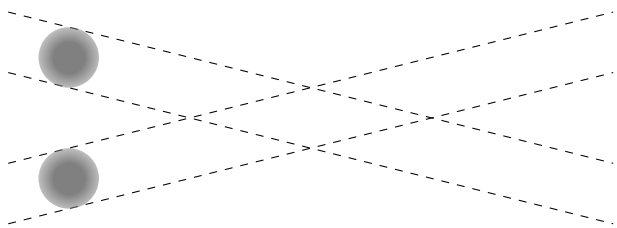
> 
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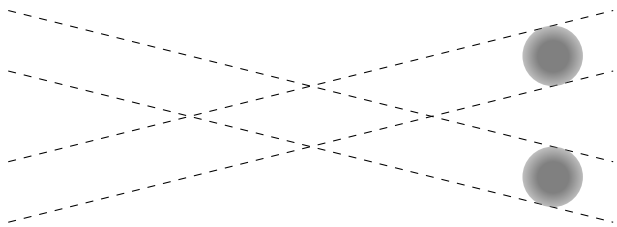
> The out-state wavefunction for large, positive t:



When there is negligible interaction, the upper picture also shows the out-state wavefunction for a large negative t, and the lower picture shows the in-state wavefunction for a large positive t, but in general these two pictures will be "messy" and describe scattering, particle production, decay, etc.

d) The time-evolution operator in the Heisenberg or Schrödinger picture is $S = e^{-iH2T}$, where H is the total Hamiltonian, and T is a large time. We know that $|\mathbf{p}\rangle$ is an eigenstate of H.

Consider an in-state of the form $|\mathbf{p}_1\mathbf{p}_2\rangle$. The following statement must be wrong, since its conclusion is that all scattering between two particles is trivial (or does not exist):

"Evolving the state $|\mathbf{p}_1\mathbf{p}_2\rangle$ from $t = -T$ to $t = T$ we obtain $e^{-iH2T}|\mathbf{p}_1\mathbf{p}_2\rangle$ in the Schrödinger picture. Since the momentum states are eigenstates of H, we have $e^{-iH2T}|\mathbf{p}_1\mathbf{p}_2\rangle = e^{-iE2T}|\mathbf{p}_1\mathbf{p}_2\rangle$, where E is the eigenvalue. Thus the evolved state is equal to the in-state multiplied by an unimportant phase factor. This means that scattering, particle production, etc. do not exist."

What is wrong?

e) Consider an interacting, scalar theory with field $\phi(x)$ in the Heisenberg picture. When $\Psi_{\mathbf{p}}(x)$ and $\Psi_{\mathbf{q}}(x)$ are spatially separated, prove that

$$\left[a^{\psi}_{\mathbf{p}}(t), a^{\psi\dagger}_{\mathbf{q}}(t)\right] = 0. \tag{2}$$

Hint: You need the equal-time commutator $[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0$.

Problem 2 Connected and amputated (J. Skaar)

a) As always you should convince yourself that you follow the derivations in the textbook and lecture notes. For example, consider Eq. (36) in the lecture note about LSZ and verify all steps, including the integration-by-parts argument (and that the surface terms vanish).

b) In the lecture notes it is argued that only the connected and amputated diagrams contribute to the nontrivial part of the S matrix elements, $i\mathcal{M}$. For the four-leg special case, argue that only the fully connected diagrams should be included when calculating $i\mathcal{M}$. An example of a diagram which is connected, but not fully connected, is given in Fig. 1.

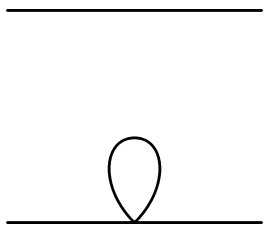


Figure 1: A four-leg diagram which is not fully connected.

Solution: For a connected four-leg diagram that is not fully connected, the external points must be pairwise connected together (possibly via a complicated system of loops). For external momenta $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{p}_1$, $\mathbf{p}_2$, this means that, for example $\mathbf{p}_1 = \mathbf{k}_1$ and $\mathbf{p}_2 = \mathbf{k}_2$. The corresponding S matrix elements make up the trivial (and uninteresting) part of S, for which the derivation of LSZ fails.

c) The diagram in Fig. 2 is not amputated. What is its $i\mathcal{M}$ matrix after amputation? What would the $i\mathcal{M}$ matrix be if you did not amputate?

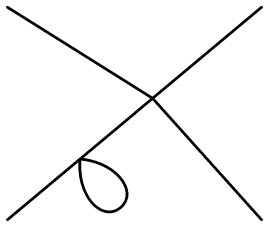


Figure 2: A fully connected four-leg diagram which is not amputated.

Solution: We use the last equation in the lecture note, with associated Feynman rules, or (4.104) in P&S pp. 114-115. With amputation:

$$i\mathcal{M} = Z^2(-i\lambda)D_F(k_1)D_F(k_2)D_F(p_1)D_F(k_1+k_2-p_1). \tag{3}$$

Without amputation:

$$\begin{aligned} i\mathcal{M} = {} & Z^2(-i\lambda)^2 D_F(k_1)D_F(k_2)D_F(p_1)D_F(k_1+k_2-p_1) \\ & \cdot \frac{1}{2}D_F(k_2)\int \frac{d^4k}{(2\pi)^4}D_F(k). \end{aligned} \tag{4}$$

Here we have a symmetry factor $S = 2$ because of the loop. The integral (which is due to the undetermined momentum in the loop) is divergent, so amputation is nice...