

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: FYS4170 — Relativistic Quantum Field Theory

Day of examination: 27. november 2023

Examination hours: 15.00 – 19.00

This problem set consists of 7 pages.

Appendices: Formula sheets are included.

Permitted aids: No examination support material is allowed.

Make sure that your copy of this examination paper is complete before answering.

Problem 1

In the course we have used ϕ^4 theory as a relatively simple entry point to many of the methods introduced for interacting quantum field theories. A theory that appears to be even simpler, which also leads to self-interactions, is ϕ^3 theory, described by the Lagrangian

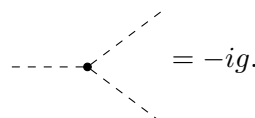
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}M^2\phi^2 - \frac{g}{3!}\phi^3. \quad (1)$$

a) Find the mass dimension n of the coupling g ($g \sim (\text{mass})^n$ in natural units).

b) The reason why we usually do not work with this theory can be seen from its Hamiltonian H . Find the expression for H , and explain why (1) cannot describe a physical theory (recall that H describes the energy of a system).

Ignoring this problem for a moment, we can take a look at the processes that are possible in this theory.

c) Determine the Feynman rules compared to those of ϕ^4 theory, and argue in particular why the interaction vertex is


$$= -ig. \quad (2)$$

You may ignore symmetry factors and combinatorics. Recall the formula

$$\langle \mathbf{p}' \mathbf{k}' | S | \mathbf{p} \mathbf{k} \rangle = {}_0 \langle \mathbf{p}' \mathbf{k}' | T e^{-i \int dt H_I(t)} | \mathbf{p} \mathbf{k} \rangle_{0 \text{ connected, amputated}}, \quad (3)$$

where “0” stands for free eigenstates, $\mathbf{p}$, $\mathbf{k}$, $\mathbf{p}'$, and $\mathbf{k}'$ are the momenta of external states, and $H_I(t)$ is the interaction part of the Hamiltonian written in the interaction picture.

d) The simplest process we can imagine is $\phi(p) \rightarrow \phi(p_1)\phi(p_2)$ (in words, this means one scalar particle in the initial state and two in the final state, with momenta in parentheses). The leading-order Feynman diagram is given by the vertex in (2). This process is usually kinematically forbidden, except for a specific combination of Lagrangian parameters and configuration of momenta p_1, p_2 . Find this combination.

e) A more interesting process is the scattering process $\phi(p_1)\phi(p_2) \rightarrow \phi(p_3)\phi(p_4)$, which is always possible. Draw all diagrams that contribute to this process to lowest order.

For the case $M = 0$, introduce Mandelstam variables,

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \quad (4)$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2, \quad (5)$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2. \quad (6)$$

where $s + t + u = 0$, and show that the spin-averaged, differential cross-section¹

$$\frac{d\sigma}{dt} = \frac{1}{32\pi s^2} \langle |\mathcal{M}|^2 \rangle, \quad (7)$$

is given by

$$\frac{d\sigma}{dt} = \frac{g^4}{32\pi} \frac{[s^2 + t^2 + st]^2}{s^4 t^2 (s + t)^2}. \quad (8)$$

Problem 2

a) The Lagrangian for the Dirac field ψ is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \quad (9)$$

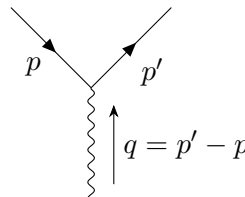
Find the Euler-Lagrange equation for ψ and $\bar{\psi}$ by treating ψ and $\bar{\psi}$ as independent.

b) The S -matrix elements are usually written in two equivalent ways:

$${}_{\text{out}}\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | \mathbf{k}_1 \mathbf{k}_2 \cdots \rangle_{\text{in}} = \langle \mathbf{p}_1 \mathbf{p}_2 \cdots | S | \mathbf{k}_1 \mathbf{k}_2 \cdots \rangle, \quad (10)$$

The reference time for the Schrödinger and Heisenberg picture is set at remote past $t = -T$. Explain what the 6 quantities $|\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle$, $|\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle_{\text{in}}$, S , $S|\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle$, $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle$, $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle_{\text{out}}$ are, relate state vectors in the two pictures, and from the relations prove the equality (10).

c) Consider a vertex in QED with an external state electron as input and as output, given at tree-level by



$$= -ie\gamma^\mu. \quad (11)$$

¹There is a factor 1/2 in this expression relative to a similar one that was derived in the exercises, which appears since the final state contains two indistinguishable particles.

Draw the lowest order vertex correction in quantum electrodynamics (QED).

The vertex correction (to any order) amounts to substitute $\gamma^\mu \mapsto \Gamma^\mu$ in the vertex rule. Argue that Γ^μ (to any order) can be written in the form

$$\Gamma^\mu = \gamma^\mu A + (p^\mu + p'^\mu)B, \quad (12)$$

where the coefficients A and B only depend on q^2 (in addition to constants such as m and e). You may assume that A is a number.

d) When calculating the cross section associated with lowest order vertex correction, it turns out that the result is infrared divergent. What is the problem, and how is it solved?

Problem 3

a) Let $\mathcal{V}$ be the set of all different, disconnected Feynman diagram pieces. Let A be the value of a particular diagram, consisting of a connected piece and (possibly) several disconnected pieces. Argue that

$$A = (\text{value of connected piece}) \cdot \prod_i \frac{1}{n_i!} V_i^{n_i}, \quad (13)$$

where V_i for $i = 1, 2, \dots$ are the (values of the) diagrams in $\mathcal{V}$. What is n_i ?

b) Prove that the sum of all diagrams with m external points is

$$\sum (\text{connected diagrams with } m \text{ external points}) \cdot \exp\left(\sum_i V_i\right), \quad (14)$$

where i indexes the elements in $\mathcal{V}$.

Formulas

Lagrangian field theory:

$$\begin{aligned}
 S &= \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \\
 \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} &= 0 \\
 \pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
 H &= \int d^3x (\pi \dot{\phi} - \mathcal{L})
 \end{aligned}$$

Noether:

$$\begin{aligned}
 \phi_r &\mapsto \phi_r + \alpha \Delta \phi_r \\
 \mathcal{L} &\mapsto \mathcal{L} + \alpha \partial_\mu J^\mu \\
 j^\mu &= \sum_r \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} \Delta \phi_r - J^\mu
 \end{aligned}$$

Lorentz transformations, boost along the z -axis:

$$\begin{aligned}
 t' &= \gamma(t + \beta z) \\
 z' &= \gamma(z + \beta t)
 \end{aligned}$$

Klein-Gordon field:

$$\begin{aligned}
 \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right) \\
 (\partial^2 + m^2)\phi &= 0 \\
 [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}).
 \end{aligned}$$

Dirac field:

$$\begin{aligned}
 \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_r \left(a_{\mathbf{p}}^r u^r(\mathbf{p}) e^{-ipx} + b_{\mathbf{p}}^{r\dagger} v^r(\mathbf{p}) e^{ipx} \right) \\
 (i\not{\partial} - m)\psi &= 0 \\
 \{a_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}\} &= \{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs} \\
 u^s(p) &= \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix}, \quad v^s(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix} \\
 \bar{u}^r(p) u^s(p) &= -\bar{v}^r(p) v^s(p) = 2m \delta^{rs} \\
 \not{p} u(p) &= m u(p), \quad \bar{u}(p) \not{p} = \bar{u}(p) m, \\
 \not{p} v(p) &= -m v(p), \quad \bar{v}(p) \not{p} = -\bar{v}(p) m \\
 \sum_{s=1}^2 u^s(p) \bar{u}^s(p) &= \not{p} + m, \quad \sum_{s=1}^2 v^s(p) \bar{v}^s(p) = \not{p} - m
 \end{aligned}$$

Chiral representation:

$$\begin{aligned}
 \gamma^0 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}; \\
 \sigma^1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
 \end{aligned}$$

Definitions:

$$\begin{aligned} f \overleftrightarrow{\partial}_0 g &= f \partial_0 g - (\partial_0 f) g \\ \sigma^\mu &\equiv (1, \sigma^i), \quad \bar{\sigma}^\mu \equiv (1, -\sigma^i) \\ \not{a} &\equiv \gamma^\mu a_\mu \text{ (4-vector } a^\mu) \end{aligned}$$

Identities:

$$\begin{aligned} \sigma^i \sigma^j &= \delta^{ij} + i \epsilon^{ijk} \sigma^k \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\ \gamma^5 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \\ \gamma^\mu \gamma_\mu &= 4 \\ \gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma_\mu &= 4g^{\nu\alpha} \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma_\mu &= -2\gamma^\rho \gamma^\alpha \gamma^\nu \\ \text{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu} \\ \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\rho} g^{\nu\sigma} + 4g^{\mu\sigma} g^{\nu\rho} \\ \text{Tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] &= 0, \text{ for } n \text{ odd} \end{aligned}$$

Ward's identity:

Let $i\mathcal{M}_\mu(q)$ be a diagram where a photon line at vertex μ is removed. Then $\mathcal{M}_\mu(q)q^\mu = 0$.

Gordon's identity:

$$\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p), \text{ where } \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \text{ and } q^\mu = p'^\mu - p^\mu.$$

Cross-sections and decay rates:

$$\begin{aligned} d\sigma &= \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_A, p_B \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A + p_B - \sum p_f) \\ d\Gamma &= \frac{1}{2m_A} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_A = (m_A, \mathbf{0}) \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A - \sum p_f) \end{aligned}$$

Two-particle phase space:

$$\int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2) = \int d\Omega_{\text{cm}} \frac{1}{16\pi^2} \frac{|\mathbf{p}_1|}{E_{\text{cm}}},$$

$E_{\text{cm}} = E_1 + E_2$ in the center-of-mass frame; Ω_{cm} = solid angle for particle 1. For decay of a single particle, set $p_B = 0$.

Lagrangians and Feynman rules

The following rules are used to obtain the amplitude $i\mathcal{M}$, and given in momentum space. Dots represent vertices in all of the diagrams listed below. For consistency between ϕ^4 theory and Yukawa theory, scalar particles are represented by dashed lines, and their masses by M , in both cases.

Common for all theories:

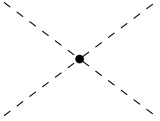
- Impose momentum conservation at each vertex
- Integrate over each undetermined momentum, $\int \frac{d^4k}{(2\pi)^4}$

 ϕ^4 theory

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

• Propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

• External line: $\xrightarrow{p} \bullet = 1$

• Vertex:  $= -i\lambda$

- Divide by the symmetry factor

Yukawa theory:

$$\mathcal{L}_Y = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2 \phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\psi\phi$$

• Scalar propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

• Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

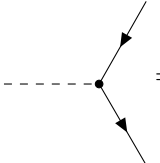
• External scalar: $\xrightarrow{p} \bullet = 1$

• Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$

• Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$

• Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$

• Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$

• Vertex:  $= -ig$

- Work out the overall sign of the diagram

QED:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad e = -|e|$$

- Photon propagator: $\mu \xrightarrow{p} \nu = \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}$
- Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon};$
- Incoming photon: $\mu \xrightarrow{p} = \epsilon_\mu(p)$
- Outgoing photon: $\bullet \xrightarrow{p} \mu = \epsilon_\mu^*(p)$
- Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$
- Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$
- Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$
- Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$
- Vertex: $\mu \xrightarrow{\quad} \bullet = -ie\gamma^\mu$
- Work out the overall sign of the diagram