University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 7

Problem 1 Dirac theory, spin, magnetization

(J. Skaar)

In this problem we consider the free, quantized Dirac theory, and show how the theory leads to an effective current density due to an inhomogeneous spin density.

The one-particle fermionic state with spin label s and momentum $\mathbf{p}$ is given by

$$|\mathbf{p}, s\rangle = \sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}}^{s\dagger}|0\rangle, \quad E_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2. \tag{1}$$

Our fermion will be expressed as

$$|\xi, s\rangle = a_\xi^{s\dagger}|0\rangle, \quad a_\xi^{s\dagger} = \int \frac{d^3p}{(2\pi)^3}\xi(\mathbf{p})a_{\mathbf{p}}^{s\dagger}, \tag{2}$$

where $a_\xi^{s\dagger}$ should be viewed as a wavepacket creation operator. The wavepacket spectrum $\xi(\mathbf{p})$ is normalized by

$$\int \frac{d^3p}{(2\pi)^3}|\xi(\mathbf{p})|^2 = 1.$$

We have used the name $\xi(\mathbf{p})$ rather than $\psi(\mathbf{p})$, since the letter ψ is used as the Dirac field.

a) Calculate $\{a_\xi^s, a_\xi^{s\dagger}\}$ and $\langle\xi, s|\xi, s\rangle = \langle 0|a_\xi^s a_\xi^{s\dagger}|0\rangle$.

> **Hint:** Use the fact that the anticommutator is linear in both inputs, and the definition of the ladder operator in (1). Both answers will be 1.

b) Show that the expectation value of the observable $\bar\psi\psi$ in the state (2) is

$$\langle\bar\psi\psi\rangle = \bar\Psi(x)\Psi(x), \tag{3}$$

up to an additive (infinite) constant. Here we have defined

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{\xi(\mathbf{p})}{\sqrt{2E_{\mathbf{p}}}}u^s(\mathbf{p})e^{-ipx}.$$

> **Hint:** The expectation value is
>
> $$\langle\bar\psi\psi\rangle = \langle 0|a_\xi^s\bar\psi\psi a_\xi^{s\dagger}|0\rangle,$$
>
> so to save some work, calculate $\psi a_\xi^{s\dagger}|0\rangle$ first, and then combine with its adjungate $\langle 0|a_\xi^s\bar\psi$. Use the result from a). The result is (3) pluss an infinite constant. (If the infinite constant bothers you, introduce a cutoff in the momentum integral.)

We now consider the nonrelativistic limit. Then the wave function $\Psi(x)$ satisfies the Schrödinger equation, as can be seen by an identical argument as that used for the scalar bosonic field. Also, since there are only two linearly independent $u^s(p)$'s (positive-frequency solutions to the Dirac equation), they span a space of two-dimensional spinors, just as in nonrelativistic quantum mechanics.

c) Show that in the nonrelativistic limit $\langle\bar{\psi}\psi\rangle$ and the charge density $\langle\bar{\psi}\gamma^0\psi\rangle$ is independent of the spin state.

Hint: For $\langle\bar{\psi}\psi\rangle$ we already have an expression $\langle\bar{\psi}\psi\rangle = \bar{\Psi}(x)\Psi(x)$; for $\langle\bar{\psi}\gamma^0\psi\rangle$ we obtain similarly $\langle\bar{\psi}\gamma^0\psi\rangle = \langle\psi^\dagger\psi\rangle = \Psi^\dagger(x)\Psi(x)$.

Approximate $u^s(p)$ in the nonrelativistic limit (see e.g. (6.36) in P&S). Verify that $\bar{\Psi}(x)\Psi(x)$ and $\Psi^\dagger(x)\Psi(x)$ are independent of the spin unit vector (called ξ^s in P&S, not to be confused with our $\xi(\mathbf{p})$).

d) Still in the nonrelativistic limit, show that the current density $\langle\bar{\psi}\gamma^i\psi\rangle$ is generally dependent on the spin state, but if the state contains a sufficiently narrow band of $\mathbf{p}$'s, the dependence is negligible.

Hint: The expectation value $\langle\bar{\psi}\gamma^i\psi\rangle = \bar{\Psi}(x)\gamma^i\Psi(x)$ involves the expression $\bar{u}^s(p')\gamma^i u^s(p)$. Then see the calculation in P&S, the two equations below (6.36), where $q = p' - p$ and you should put $\xi'^s = \xi^s$. What happens if $\xi(\mathbf{p})$ tends to a delta function $\delta(\mathbf{p} - \mathbf{p}_0)$?

e) Prove that the spin dependent part of the current density $\langle\bar{\psi}\boldsymbol{\gamma}\psi\rangle$ can be written $\nabla \times \mathbf{M}$, where $\mathbf{M} = \Psi^\dagger\boldsymbol{\sigma}\Psi = g\Psi^\dagger\frac{\boldsymbol{\sigma}}{2}\Psi$, and Ψ is defined as before, but with the Dirac-spinor $u^s(\mathbf{p})$ replaced by the two-component spinor ξ^s. The parameter $g = 2$ is the Landé g-factor (which will be calculated more accurately in Ch. 6 of P&S).

Give a physical interpretation.

Hint: For the interpretation, remember your electromagnetics (or magnetostatics). See e.g. example 3.13 and 3.14 in `https://www.mn.uio.no/fysikk/personer/vit/johask/elektromagnetisme.pdf`, p. 87-91 (in Norwegian), or Wikipedia "magnetization".

Problem 2 Schrödinger, Heisenberg, interaction picture **(J. Skaar)**

We restrict our attention to Hamiltonians independent of time, which is the case for closed systems. In the Schrödinger picture, which is commonly used in nonrelativistic quantum mechanics, the states evolve according to the Schrödinger equation,

$$i\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle, \tag{4}$$

while the operators are independent of time. Since H is independent of time, the Schrödinger equation can be solved easily:

$$|\psi(t)\rangle = e^{-iH(t-t_0)}|\psi(t_0)\rangle. \tag{5}$$

We define t_0 as a fixed reference time, where all three pictures (Schrödinger, Heisenberg, interaction) coincide by definition. Thus for $t = t_0$, the state vector is $|\psi(t_0)\rangle$ and the operator A is $A(t_0)$ in all pictures.

a) *Heisenberg picture:* In the Heisenberg picture, the state vector $|\psi(t_0)\rangle$ is independent of time, while the operators evolve. Given that the expectation values $\langle A(t)\rangle$ must be the same in all pictures, argue that

$$A(t) = e^{iH(t-t_0)}A(t_0)e^{-iH(t-t_0)} \tag{6}$$

in the Heisenberg picture. Also prove the Heisenberg equation of motion

$$i\frac{dA(t)}{dt} = [A(t), H]. \tag{7}$$

b) *Interaction picture:* Let us write the Hamiltonian in the form $H = H_0 + H_{\text{int}}$. Let $|\psi(t)\rangle_I$ and $A_I(t)$ be the state vector and an operator in the interaction picture. The interaction picture is related to the Schrödinger picture by

$$A_I(t) = e^{iH_0(t-t_0)}Ae^{-iH_0(t-t_0)}, \tag{8a}$$

$$|\psi(t)\rangle_I = e^{iH_0(t-t_0)}|\psi(t)\rangle, \tag{8b}$$

Argue that in the interaction picture, the time dependence due to H_0 is associated with the operators, and prove that

$$i\frac{dA_I(t)}{dt} = [A_I(t), H_0]. \tag{9}$$

Prove that

$$|\psi(t)\rangle_I = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}|\psi(t_0)\rangle, \tag{10}$$

and argue that in the special case where H_0 and H_{int} commute, the time dependence due to H_{int} is associated with the state vector.

c) In P&S the unitary operator in (10) is called the time evolution operator $U(t, t_0)$:

$$U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}. \tag{11}$$

Here it is important to note that t_0 is not arbitrary; it is the reference time where the different pictures coincide. We want a unitary operator $U(t, t')$ to describe the evolution of state vectors in the interaction picture, from any time t' to t. We must require

$$U(t, t') = U^\dagger(t', t) \tag{12a}$$

$$U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1). \tag{12b}$$

Use (11) and (12) to prove that

$$U(t, t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)}. \tag{13}$$

Hint: Put $t_3 = t$, $t_1 = t'$, and $t_2 = t_0$.

d) While the Hamiltonian H is the same (and independent of time) in the Schrödinger and Heisenberg pictures, verify that the Hamiltonian in the interaction picture can be written $H_0 + H_I(t)$, where $H_I(t)$ is dependent on time:

$$H_I(t) = e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)}, \tag{14}$$

e) Prove that the connection between an Heisenberg operator $A(t)$ and the operator in the interaction picture is

$$A(t) = U^\dagger(t, t_0) A_I(t) U(t, t_0). \tag{15}$$