

Feynman rules for fermions

Recall : $T \psi(x) \bar{\psi}(y) = \begin{cases} \psi(x) \bar{\psi}(y), & x^0 > y^0, \\ -\bar{\psi}(y) \psi(x), & x^0 < y^0. \end{cases}$

$$S_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} = \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle$$

More than two fields: $T(\psi(x_1) \psi(x_2) \psi(x_3) \psi(x_4)) = (-1)^3 \psi(x_3) \psi(x_1) \psi(x_4) \psi(x_2)$
if $x_3^0 > x_1^0 > x_4^0 > x_2^0$.

Normal ordering : $N(a_p a_q a_p^\dagger) = (-1)^2 a_p^\dagger a_p a_q = (-1)^3 a_p^\dagger a_q a_p$
(minus sign for each fermion interchange).

Wick's theorem : $T \psi(x) \bar{\psi}(y) = N \psi(x) \bar{\psi}(y) + \overline{\psi(x) \bar{\psi}(y)}$

If this is taken as the def. of $\overline{}$ we have:

$$\begin{aligned} \overline{\psi(x) \bar{\psi}(y)} &= \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = S_F(x-y) \\ &= \begin{cases} \{\psi(x), \bar{\psi}(y)\}, & x^0 > y^0 \\ -\{\bar{\psi}(y), \psi(x)\}, & x^0 < y^0 \end{cases} = -\overline{\bar{\psi}(y) \psi(x)} \end{aligned}$$

exercise

$$\overline{\psi(x) \psi(y)} = 0 = \overline{\bar{\psi}(x) \bar{\psi}(y)}$$

Define contractions under $N\{\}$ to include - for operator interchanges:

$$N \overline{\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4} = -\overline{\psi_1 \bar{\psi}_3} N \psi_2 \bar{\psi}_4 = -S_F(x_1 - x_3) N \psi_2 \bar{\psi}_4$$

Wick : $T \psi_1 \bar{\psi}_2 \psi_3 \dots = N[\psi_1 \bar{\psi}_2 \psi_3 \dots + \text{all possible contractions}]$

When there is a bosonic field ϕ , use that $[\phi, \psi] = 0 = [\phi^\dagger, \psi^\dagger] = [\phi^-, \psi^\dagger]$
so $\overline{\phi \psi} = 0$.

Yukawa theory

int. prob. $H_I(t) = \int d^3x g \bar{\psi}_I \psi_I \phi_I$

$$H = H_{\text{Dirac}} + H_K + \int d^3x g \bar{\psi} \psi \phi$$

Consider $\langle \bar{p}' h' | S | \bar{p} h \rangle = \langle \bar{p}' h' | T e^{-i \int dt H_I(t)} | \bar{p} h \rangle_0$ connected, amputated
two ferm. 1 2 *two fermions in*

0th order: trivial / forward scattering

1st order: 0 (why)

2th order: $\langle \bar{p}' h' | T \frac{1}{2!} (-ig) \int d^4x \bar{\psi} \psi \phi \quad (-ig) \int d^4y \bar{\psi} \psi \phi \quad | \bar{p} h \rangle_0$ (*)

$\overline{\psi(x) | \bar{p}, s \rangle_0} = \overline{\psi(x) | \bar{p}, s, -\infty \rangle}$, where $\bar{\psi}_{\vec{p}, s}(-\infty) = \sqrt{2E_{\vec{p}}} (b_{\vec{p}}^s + a_{\vec{p}}^{s\dagger})$
antifield "field" that creates $|\bar{p}, s\rangle$:
 $\bar{\psi}_{\vec{p}, s}(-\infty) | 0 \rangle = |\bar{p}, s \rangle$
 $\{ \psi^+(x), \sqrt{2E_{\vec{p}}} a_{\vec{p}}^{s\dagger} \}$

$$\left\{ \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_r a_r^s u^r(q) e^{-iqx}, \sqrt{2E_p} a_p^{s\dagger} \right\} = u^s(p) e^{-ipx}$$

$$\left[\begin{aligned} \overline{\psi(x) | \bar{p}, s, -\infty \rangle} &= (T - N) \overline{\psi(x) | \bar{p}, s, -\infty \rangle} \\ &= \psi \bar{\psi}_{\vec{p}, s} - \psi^- \bar{\psi}_{\vec{p}, s}^+ + \bar{\psi}_{\vec{p}, s} \psi^+ - \psi^- \bar{\psi}_{\vec{p}, s}^- - \psi^+ \bar{\psi}_{\vec{p}, s}^+ \\ &= \psi^+ \bar{\psi}_{\vec{p}, s} + \bar{\psi}_{\vec{p}, s} \psi^+ = \{ \psi^+(x), \bar{\psi}_{\vec{p}, s}(-\infty) \} \end{aligned} \right]$$

Note that x is not an external point, it is a vertex.

So the e^{-ipx} is used to obtain a mom. conserving δ -function, after integration wrt. x .

A contribution to (*):

$$\langle \bar{p}' h' | T \frac{1}{2!} (-ig) \int d^4x \bar{\psi} \psi \phi \quad (-ig) \int d^4y \bar{\psi} \psi \phi \quad | \bar{p} h \rangle_0$$

omit this since there is an identical term with $x \leftrightarrow y$.

Contractions $\overline{\psi \psi}$, $\overline{\psi \phi}$, $\langle \bar{h}' | \psi$, $\overline{\psi | h} \rangle_0$ vanish.

Diagram:

$$= (-ig)^2 \frac{i}{q^2 - m_\phi^2 + i\epsilon} \bar{u}(p') u(p) \bar{u}(h') u(h), \quad q = p - p'$$

Feynman rules for iM , Yukawa $H_{int} = \int d^3x g \bar{\psi} \psi \phi$

1. Propagators: $\overline{\phi(x)} \phi(y) = \text{---} \overrightarrow{x} \text{---} = \frac{i}{q^2 - m_\phi^2 + i\epsilon} = D_F(q)$

$\overline{\psi(x)} \psi(y) = \text{---} \overrightarrow{p} \text{---} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} = S_F(p)$

2. Vertices:  $= (-ig)$

3. External leg contractions:

$\overline{\phi} | \vec{q} \rangle = \text{---} \overrightarrow{q} \text{---} = 1$

$\langle \vec{q} | \phi = \text{---} \overrightarrow{q} \text{---} = 1$

$\overbrace{\psi | \vec{p}, s \rangle}^{\text{fermion}} = \text{---} \overrightarrow{p} \text{---} = u^s(p)$

$\underbrace{\langle \vec{p}, s | \bar{\psi}}_{\text{fermion}} = \text{---} \overrightarrow{p} \text{---} = \bar{u}^s(p)$

$\overbrace{\bar{\psi} | \vec{k}, s \rangle}^{\text{anti-ferm}} = \text{---} \overleftarrow{k} \text{---} = \bar{v}^s(k)$

$\underbrace{\langle \vec{k}, s | \psi}_{\text{anti-ferm}} = \text{---} \overleftarrow{k} \text{---} = v^s(k)$

4. Mom. conservation at each vertex.

5. Integrate over undetermined loop momenta.

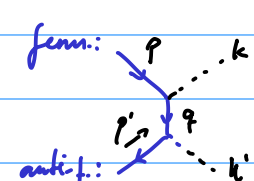
6. Figure out overall sign of the diagram due to fermion field interchanges.

Comments:

▣ No symmetry factors: Propagators have direction ($S_F(p) \neq S_F(p)$).

Two equal lines cannot end at the same vertex: $\int d^3x g \bar{\psi} \psi \phi$

▣ Convention for arrows:

 $= (-ig)^2 \bar{v}^s(p') S_F(q) u^s(p), \quad q = p - k = k' - p'$

▣ Fermion minus signs:

$\langle \vec{p}' \vec{k}' | \dots (\bar{\psi} \psi)_x \dots (\bar{\psi} \psi)_y \dots | \vec{p} \vec{k} \rangle, \quad |\vec{p} \vec{k}\rangle \stackrel{\text{def}}{\sim} a_{\vec{p}}^\dagger a_{\vec{k}}^\dagger |0\rangle$

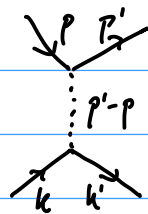
$= \langle 0 | a_{\vec{k}} a_{\vec{p}} \dots (\bar{\psi} \psi)_x \dots (\bar{\psi} \psi)_y \dots a_{\vec{p}}^\dagger a_{\vec{k}}^\dagger |0\rangle \quad (1)$

Sign: $(-1)^2$.

$\langle 0 | a_{\vec{k}} a_{\vec{p}} \dots (\bar{\psi} \psi)_x \dots (\bar{\psi} \psi)_y \dots a_{\vec{p}}^\dagger a_{\vec{k}}^\dagger |0\rangle \quad (2)$

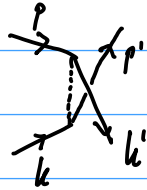
Sign: (-1)

(1):



$$= (-ig)^2 D_F(p'-p) \bar{u}(p') u(p) \bar{u}(k') u(k)$$

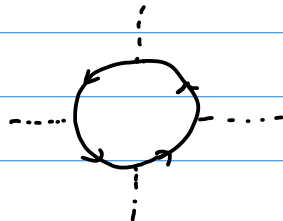
(2):



$$= -(-ig)^2 D_F(p'-k) \bar{u}(k') u(p) \bar{u}(p') u(k)$$

$$iM = (1) + (2).$$

□ Closed fermion loops:



$$\begin{aligned} &: \overbrace{\bar{\psi} \psi \bar{\psi} \psi \bar{\psi} \psi \bar{\psi} \psi} \\ &= \bar{\psi} S_F S_F S_F \psi \\ &= \bar{\psi}_\alpha S_{F\alpha\beta} S_{F\beta\gamma} S_{F\gamma\delta} \psi_\delta \\ &= - S_{F\alpha\beta} S_{F\beta\gamma} S_{F\gamma\delta} S_{F\delta\alpha} \\ &= -\text{tr}(S_F S_F S_F S_F) \end{aligned}$$

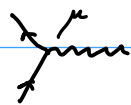
(closed fermion loops give (-1) and tr .)

Feynman rules QED

$$H_{int} = \int d^3x e \bar{\psi} \gamma^\mu \psi A_\mu$$

Addition to the fermion rules:

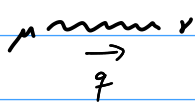
New vertex:



$$= -ie \gamma^\mu$$

Yukawa: $(-ig)$
QED: $(-ie) \gamma^\mu$

Photon prop.:



$$= \frac{-ig_{\mu\nu}}{q^2 + i\epsilon}$$

Problem set 6

External photon lines:

$$\overbrace{A_\mu | \vec{k}, r \rangle}^{\text{polarization}} = \overbrace{\text{wavy line}}^{\vec{k}}_\mu = \epsilon_\mu^r(k)$$

$$\overbrace{\langle \vec{k}, r | A_\mu} = \overbrace{\text{wavy line}}^\mu_{\vec{k}} = \epsilon_\mu^{r*}(k)$$

Same arguments
as for KG or
fermions.

$$\epsilon_\mu^0(k) = (1, \vec{0}), \quad \epsilon_\mu^3(k) = \vec{k}/|\vec{k}|, \quad \epsilon_\mu^1(k) \text{ and } \epsilon_\mu^2(k) \perp \vec{k}.$$

The Yukawa potential and the Coulomb potential

Example: Scattering of distinguishable fermions, nonrel. limit.

Yukawa:

$$iM = \begin{array}{c} \text{Diagram: Incoming fermions } p, p' \text{ and outgoing } k, k' \text{ connected by a scalar exchange line } p'-p. \end{array} = (-ig)^2 \bar{u}(p') u(p) \frac{i}{q^2 - m_\phi^2 + i\epsilon} \bar{u}(k') u(k)$$

QED:

$$iM = \begin{array}{c} \text{Diagram: Incoming fermions } p, p' \text{ and outgoing } k, k' \text{ connected by a photon exchange line } q. \end{array} = (-ie)^2 \bar{u}(p') \gamma^\mu u(p) \frac{-ig_{\mu\nu}}{q^2 + i\epsilon} \bar{u}(k') \gamma^\nu u(k)$$

[The "twin diagram" ~~does~~ does not contribute since the fermions are distinguishable.]

To first order in 3-momentum: $E_p^2 = m^2 + \vec{p}^2 = m^2$

$p = (m, \vec{p})$, $p' = (m, \vec{p}')$, $k = (m, \vec{k})$, $k' = (m, \vec{k}')$. [Assume m the same for the two particles]

$$(p' - p)^2 = (m - m)^2 - (\vec{p}' - \vec{p})^2 = -(\vec{p}' - \vec{p})^2, \quad p \cdot \sigma = m1 - \vec{p} \cdot \vec{\sigma} = m$$

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}$$

$$\bar{u}^{s'}(p') u^s(p) = m \begin{pmatrix} \xi^{s' \dagger} & \xi^{s' \dagger} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} = 2m \delta^{ss'}, \quad \bar{u}^{r'}(k') u^r(k) = 2m \delta^{rr'}$$

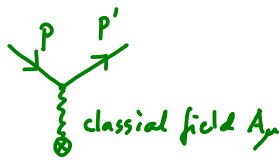
$$\bar{u}^{s'}(p') \gamma^0 u^s(p) = 2m \delta^{ss'}, \quad \bar{u}^{r'}(k') \gamma^0 u^r(k) = 2m \delta^{rr'}$$

$$\bar{u}^{s'}(p') \gamma^i u^s(p) = m \begin{pmatrix} \xi^{s' \dagger} & \xi^{s' \dagger} \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} = 0 \quad (\text{that is, } \rightarrow 0 \text{ for } p, p' \rightarrow 0).$$

$$iM_Y = \frac{ig^2}{(\vec{p}' - \vec{p})^2 + m_\phi^2} 2m \delta^{ss'} 2m \delta^{rr'}, \quad iM_{QED} = \frac{-ie^2}{(\vec{p}' - \vec{p})^2 - i\epsilon} 2m \delta^{ss'} 2m \delta^{rr'}$$

• Opposite signs

• $iM = 0$ unless $s' = s$, $r' = r$, so spin of each particle is separately conserved.



From nonrel quantum mech.: $\langle p' | iT | p \rangle = -i V(\vec{q}) 2\pi \delta(E_{p'} - E_p)$, $\vec{q} = \vec{p}' - \vec{p}$.

(or Problem 4.4 p. 129; Pr. set 10, 2b)

Define an $\mathcal{M}$: $\langle p' | iT | p \rangle = i\mathcal{M} 2\pi \delta(E_{p'} - E_p)$

(Turns out that this one-particle- $\mathcal{M}$ can be compared to the previous $\mathcal{M}$'s except the normalization factors. Can be shown by relating the cross sections in the two descriptions.)

Yukawa: $V(\vec{q}) = -\frac{g^2}{\vec{q}^2 + m_\pi^2} \Rightarrow V(\vec{r}) = \mathcal{F}^{-1} V(\vec{q}) = -\frac{g^2}{4\pi r} e^{-m_\pi r}$

QED: $V(\vec{q}) = \frac{e^2}{\vec{q}^2} \Rightarrow V(\vec{r}) = \frac{e^2}{4\pi r}$

Attractive Yukawa-pot.,
range $\frac{1}{m_\pi} = \frac{\hbar}{m_\pi c}$

(analogous w. exchanged boson)
Repulsive QED-pot./
Coulomb-pot.

Yukawa: $ff, f\bar{f}, \bar{f}\bar{f}$: attractive

QED: $f\bar{f}$ attractive, $ff, \bar{f}\bar{f}$ repulsive