

University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 10

Problem 1 Contractions for fermion fields (J. Skaar)

a) Wick's theorem for two fermion fields is

$$T\psi(x)\bar{\psi}(y) = N\psi(x)\bar{\psi}(y) + \overline{\psi(x)\bar{\psi}(y)}. \quad (1)$$

If (1) is taken as the definition of the contraction $\overline{\psi(x)\bar{\psi}(y)}$, and

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle, \quad (2)$$

prove that

$$\begin{aligned} \overline{\psi(x)\bar{\psi}(y)} &= \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = S_F(x-y) \\ &= \begin{cases} \{\psi^+(x), \bar{\psi}^-(y)\}, & x^0 > y^0, \\ -\{\bar{\psi}^+(y), \psi^-(x)\}, & x^0 < y^0, \end{cases} \\ &= -\overline{\bar{\psi}(y)\psi(x)}, \end{aligned} \quad (3)$$

and

$$\overline{\psi(x)\psi(y)} = 0 = \overline{\bar{\psi}(x)\bar{\psi}(y)} \quad (4)$$

(Strictly speaking, we need spinor indices on the spinors above to make sense of the equations).

Solution: The first equality follows by taking the vacuum expectation value of (1), since the vacuum expectation value of normal ordered operators is zero. For the last equality, note that for $x^0 > y^0$,

$$\begin{aligned} T\psi(x)\bar{\psi}(y) &= \psi(x)\bar{\psi}(y) = \psi^+(x)\bar{\psi}^+(y) + \psi^+(x)\bar{\psi}^-(y) \\ &\quad + \psi^-(x)\bar{\psi}^+(y) + \psi^-(x)\bar{\psi}^-(y). \end{aligned} \quad (5)$$

Taking the vacuum expectation value (and taking into account the usual expressions for $\psi(x)$ and $\bar{\psi}(y)$ in terms of ladder operators), we obtain

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \langle 0|\psi^+(x)\bar{\psi}^-(y)|0\rangle = \{\psi^+(x), \bar{\psi}^-(y)\}. \quad (6)$$

When $x^0 < y^0$ we have $T\psi(x)\bar{\psi}(y) = -\bar{\psi}(y)\psi(x)$, and proceed in the same way.

b) For all examples in P&S (see e.g. the section about Yukawa theory in Ch. 4), none of them has contractions between two external states (between two initial-state particles, between two final-state particles, or between initial-state and final-state particles). Can you explain why this possibility is ignored?

c) Consider a process involving two different fermions, e.g., an electron and a muon. In the QED Lagrangian we then have two independent fermionic fields ψ_e and ψ_μ , both of them coupled to the electromagnetic field. That the fermionic fields are independent means that $0 = \{\psi_e, \psi_\mu\} = \{\psi_e, \bar{\psi}_\mu\} = \{\psi_\mu, \bar{\psi}_e\}$ (spinor indices implied). Also ladder operators associated with independent fields anti-commute¹.

For QED, can the different parts of a single fermion line in a Feynman diagram refer to different fermions? Justify the answer.

Solution: No.

The S matrix consists of terms of the form

$$\begin{aligned} & \langle \mathbf{p}' \mathbf{k}' \dots | S | \mathbf{p} \mathbf{k} \dots \rangle \\ & \propto {}_0 \langle \mathbf{p}' \mathbf{k}' \dots | T \int d^4x \bar{\psi}_e \gamma^\mu \psi_e A_\mu \int d^4y \bar{\psi}_\mu \gamma^\nu \psi_\mu A_\nu \dots | \mathbf{p} \mathbf{k} \dots \rangle_0, \end{aligned} \quad (7)$$

where the right-hand side is limited to those terms that correspond to connected and amputated diagrams.

First of all, from the form of each integrand, we note that a vertex connects only lines of the same particle type.

The different parts of a fermion line origin from contractions, both between fields, or between fields and external states. Contractions between different fermionic fields vanish, since their associated ladder operators anti-commute. Similarly, contractions between a field ψ_e and an external state $|\mathbf{p}\rangle_\mu$ associated with the other field must vanish.

Thus all parts of a single fermion line refer to a given field, either the muon field or the electron field.

Problem 2 Interaction between fermions and a classical electromagnetic field

a) Peskin & Schroeder problem 4.4a p. 129. Note that your result can be represented as a Feynman diagram with associated rule (see 4.4b).

Solution: Write $S = 1 + iT$ (and always keep the i in iT to make it different from the time ordering symbol T):

$$\begin{aligned} \langle p' | iT | p \rangle &= {}_0 \langle p' | T e^{-i \int dt H_I(t)} | p \rangle_{0, \text{connected, amputated}} \\ &= {}_0 \langle p' | T e^{-i \int d^4x e \bar{\psi} \gamma^\mu \psi A_\mu} | p \rangle_{0, \text{connected, amputated}}. \end{aligned} \quad (8)$$

Here we have used

$$H_I(t) = \int d^3x e \bar{\psi} \gamma^\mu \psi A_\mu, \quad (9)$$

where A_μ represent the classical, electromagnetic field, and ψ is the interaction-picture (quantized) Dirac field.

¹It is also possible to choose the ladder operators (and fields) of independent fields to commute rather than anti-commute. See <https://arxiv.org/pdf/1312.0831>.

To first order:

$$\begin{aligned}
 \langle p' | iT | p \rangle &= -ie \int d^4x A_\mu \cdot {}_0 \langle p' | T \bar{\psi} \gamma^\mu \psi | p \rangle_0 = -ie \int d^4x A_\mu \cdot {}_0 \langle p' | \overline{\psi} \gamma^\mu \psi | p \rangle_0 \\
 &= -ie \int d^4x A_\mu(x) e^{i(p'-p)x} \bar{u}(p') \gamma^\mu u(p) \\
 &= -ie \tilde{A}_\mu(p' - p) \bar{u}(p') \gamma^\mu u(p). \tag{10}
 \end{aligned}$$

To obtain the second line, we use that the contraction between $\psi(x)$ and $|p\rangle_0$ gives $u(p)e^{-ipx}$. The other contraction gives $\bar{u}(p')e^{ip'x}$. Note the presence of the exponentials. (In the momentum-space Feynman rules for QED or Yukawa theory, the exponentials are omitted, but that is because they have been used together with $\int d^4x$ to obtain momentum-conserving delta functions.)

We note that (10) can be represented by the Feynman diagram and Feynman rule mentioned in 4.4b.

b) Specialize your result to the case where the classical field is a scalar potential $V(\mathbf{x})$ which is independent of time. Also assume that the particle is nonrelativistic.

Hint: You need the equation below (4.133) in P&S.

Solution: To get a nonzero result, the input and output spins must match. The result is

$$\langle p' | iT | p \rangle = -i\tilde{V}(\mathbf{q}) 2\pi\delta(E_{\mathbf{p}'} - E_{\mathbf{p}}) \cdot 2m, \quad \mathbf{q} = \mathbf{p}' - \mathbf{p}, \tag{11}$$

which is (4.124) in P&S, except for the extra $2m$ factor due to the relativistic normalization of $|p\rangle$ and $|p'\rangle$ here.

c) In what sense is 4-momentum conserved in this process, in the general case and in the special case where the classical field is independent of time?

Solution: We have $p' = p + q$, i.e., the output momentum is equal to the input momentum plus the momentum from the Fourier component of the classical field. In the general case the classical field consists of a spectrum of Fourier components with momenta q ; this leads to a spectrum of output momenta p' even though p is fixed. When the classical field is independent of time the Fourier transform contains a delta function $\delta(q^0)$ so the input and output energies of the fermion are equal.

Problem 3 Spinor algebra and trace methods (T. Klungland)

a) For any operator Γ consisting of a product of an arbitrary number of γ matrices, and any two spinors $v^s(p)$, $u^r(k)$ (r and s label the spin states of the two spinors) (whether the spinors are particle or antiparticle spinors is irrelevant; these are just chosen as an example), show that

$$(\bar{v}^s(p)\Gamma u^r(k))^\dagger = \bar{u}^r(k)\Gamma' v^s(p), \quad (12)$$

where $\bar{u} = u^\dagger \gamma^0$ and Γ' is the same product as Γ with the order of matrices reversed. You will need the identity $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$.

Solution: Writing Γ as $\Gamma = \gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_n}$, its hermitian conjugate is given by

$$\begin{aligned} \Gamma^\dagger &= (\gamma^{\mu_n})^\dagger \dots (\gamma^{\mu_2})^\dagger (\gamma^{\mu_1})^\dagger \\ &= \gamma^0 \gamma^{\mu_n} \gamma^0 \dots \gamma^0 \gamma^{\mu_2} \gamma^0 \gamma^0 \gamma^{\mu_1} \gamma^0 \\ &= \gamma^0 \gamma^{\mu_n} \dots \gamma^{\mu_2} \gamma^{\mu_1} \gamma^0 \\ &= \gamma^0 \Gamma' \gamma^0, \end{aligned}$$

where the third equality follows since all “interior” γ^0 matrices are in pairs, all of which multiply to 1. Using this, our desired result follows almost immediately:

$$\begin{aligned} (\bar{v}^s(p)\Gamma u^r(k))^\dagger &= \left(v^{s\dagger}(p) \gamma^0 \Gamma u^r(k) \right)^\dagger \\ &= u^{r\dagger}(k) \Gamma^\dagger \gamma^0 v^s(p) \\ &= \bar{u}^r(k) \Gamma' v^s(p). \end{aligned}$$

b) Suppose that the scattering of a fermion off of some potential is described by the matrix element

$$i\mathcal{M} = iB_\mu \bar{u}^r(p') \gamma^\mu u^s(p),$$

where p and p' describe the initial and final momenta, respectively, and s and r the initial and final spins. B_μ contains the other numerical factors, that are unimportant for our purposes. Use the spin sum relation in Eq. (3.66) in P&S to show that the unpolarized, spin-averaged squared matrix element (this means that we average over the 2 possible initial-state spins and sum over the final-state ones) is given by

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{2} \sum_{s=1}^2 \sum_{r=1}^2 [B_\mu \bar{u}^r(p') \gamma^\mu u^s(p)]^\dagger [B_\nu \bar{u}^r(p') \gamma^\nu u^s(p)] \quad (13)$$

$$= \frac{1}{2} B_\mu^* B_\nu \text{Tr}[(\not{p}' + m) \gamma^\mu (\not{p} + m) \gamma^\nu], \quad (14)$$

where m is the mass of the fermion and $\not{p} = p_\mu \gamma^\mu$.

Hint: One way of doing this is explained on p. 132 in P&S, but it is easier to use a) that a product of matrices that gives a scalar, or a 1×1 matrix can be written as its own trace, and b) the cyclic and linear properties of traces.

Solution: Using the result of part a, we first have

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{2} B_\mu^* B_\nu \sum_{s=1}^2 \sum_{r=1}^2 \bar{u}^s(p) \gamma^\mu u^r(p') \bar{u}^r(p') \gamma^\nu u^s(p).$$

Next we note that $\bar{u}^s(p) \gamma^\mu u^r(p') \bar{u}^r(p') \gamma^\nu u^s(p) = \text{Tr}[\text{same expression}]$, and use the cyclic property of traces to move $u^s(p)$ from the back to the front; moving the sums into the trace (which we can do because of the linearity of traces) and using the aforementioned spin sum relation, we find

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{1}{2} B_\mu^* B_\nu \text{Tr} \left[\left(\sum_{s=1}^2 u^s(p) \bar{u}^s(p) \right) \gamma^\mu \left(\sum_{r=1}^2 u^r(p') \bar{u}^r(p') \right) \gamma^\nu \right] \\ &= \frac{1}{2} B_\mu^* B_\nu \text{Tr} [(\not{p}' + m) \gamma^\mu (\not{p} + m) \gamma^\nu]. \end{aligned}$$