University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 6

Problem 1 The photon propagator (J. Skaar)

The purpose of this problem is to get experience with classical field theory, be aware of some of the subtleties in quantizing the electromagnetic field, and in the end obtain the photon Feynman propagator.

a) The Lagrangian for the Maxwell field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - j_\mu(x)A^\mu(x), \tag{1}$$

where $j^\mu(x)$ is the source or four-current density, and

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x). \tag{2}$$

The two Maxwell equations without sources (Faraday's law and Gauss' law for $\mathbf{B}$) follow from (2).

We now treat the four components of the potential $A^\mu(x)$ as the independent, dynamical variables. Show that the Euler-Lagrange equations with (1) lead to Maxwell's equations with sources,

$$\partial_\nu F^{\mu\nu}(x) = -j^\mu(x). \tag{3}$$

Hint: The Lagrangian can be written

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - j_\mu A^\mu, \\
&= -\frac{1}{4}[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu) - (\partial_\nu A_\mu)(\partial^\mu A^\nu) + (\partial_\nu A_\mu)(\partial^\nu A^\mu)] - j_\mu A^\mu, \\
&= -\frac{1}{2}[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)] - j_\mu A^\mu, \\
&= -\frac{g^{\mu\rho}g^{\nu\sigma}}{2}[(\partial_\mu A_\nu)(\partial_\rho A_\sigma) - (\partial_\mu A_\nu)(\partial_\sigma A_\rho)] - j_\mu A^\mu.
\end{aligned}$$

Then calculate $\frac{\partial \mathcal{L}}{\partial(\partial_\kappa A_\lambda)}$.

Solution: From the hint we can calculate

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\kappa A_\lambda)} &= -\frac{1}{2}\Big[g^{\kappa\rho}g^{\lambda\sigma}\partial_\rho A_\sigma + g^{\mu\kappa}g^{\nu\lambda}\partial_\mu A_\nu - g^{\kappa\rho}g^{\lambda\sigma}\partial_\sigma A_\rho - g^{\mu\lambda}g^{\nu\kappa}\partial_\mu A_\nu\Big] \\
&= \partial^\lambda A^\kappa - \partial^\kappa A^\lambda.
\end{aligned} \tag{4}$$

The Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial A_\nu}$$

becomes

$$\partial_\mu(\partial^\nu A^\mu - \partial^\mu A^\nu) = -j^\nu \tag{5}$$

or

$$\partial_\mu F^{\nu\mu} = -j^\nu.$$

b) What is the canonical momenta π_μ corresponding to the fields $A^\mu(x)$? Note that $\pi_0 = 0$.

Solution: The canonical momenta are defined by $\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu}$. From (4) we obtain $\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = \partial^\mu A^0 - \partial^0 A^\mu = F^{\mu 0}$. Thus $\pi^i = E^i$ (the electric field), while $\pi^0 = 0$.

c) In the canonical quantization procedure we impose commutator relations

$$[A^\mu(x), \pi_\nu(y)] = i\delta^\mu_\nu \delta^{(3)}(x-y). \tag{6}$$

Argue that this cannot be done for the Lagrangian treated so far.

Hint: Consider $\nu = 0$.

The difficulty with the Lagrangian (1) as a starting point for quantization is that the four components of A^μ are not independent. First of all, the scalar potential A^0 is not an independent dynamical variable, which is seen from the fact that $F^{\mu\nu}$ and therefore the Lagrangian do not contain $\dot{A}^0$. Put differently, it turns out that A^0 is determined from $\nabla \cdot \mathbf{A}$ and the charge density ρ, as can be shown from Gauss' law $\nabla \cdot \mathbf{E} = \rho$, with $\mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t}$.

Also, as is well known from classical electromagnetics we may choose a gauge without altering the gauge-independent electric and magnetic fields. For example, we may choose the Lorenz gauge

$$\partial_\mu A^\mu = 0. \tag{7}$$

Then the field equations (3) become

$$\Box A^\mu = j^\mu. \tag{8}$$

Two common strategies for canonical quantization of the electromagnetic field, is to employ the Lorenz gauge $\partial_\mu A^\mu = 0$ or the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. The latter gauge gives $A^0 = 0$ and that $\mathbf{A}$ is transversal (i.e., in the Fourier representation we have $\mathbf{k} \cdot \mathbf{A}(\mathbf{k}) = 0$, found from $\nabla \cdot \mathbf{A}(x) = 0$). Then only two degrees of freedom remain, corresponding to the two-dimensional space of possible polarizations. After quantization we then obtain tranverse photons. This is the method used in nonrelativistic quantum optics (e.g. in FYS4110), but sometimes also in relativistic QFT. One disadvantage with the Coulomb gauge is that Lorentz invariance gets hided.

Therefore we will rather use the Lorenz gauge. Since in the Lorenz gauge, we have the field equations (8), we will rather use a Lagrangian leading directly to (8):

$$\mathcal{L} = -\frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) - j_\mu A^\mu. \tag{9}$$

Then we have in fact modified the theory, but that's ok as long as we remember the Lorenz condition (7), making the two sets of field equations equivalent.

From now on, we will mostly consider the source-free case $j^\mu = 0$. For each field component A^μ the field equation (8) is the same as the massless

Klein-Gordon equation. However, by comparing the Lagrangian to Eq. (2.6) in P&S, we find that the terms with A^i have the same sign as in the Klein-Gordon case, while the term with A^0 has opposite sign. The canonical momenta are $\pi_\mu = -\dot{A}_\mu$, i.e., $-\dot{A}^0$ and $\dot{A}^i$. Going through the math in the same way as for the Klein-Gordon field, one therefore obtains the Feynman propagator

$$D_F^{\mu\nu}(x-y) = \langle 0| T A^\mu(x) A^\nu(y) |0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\mu\nu}}{k^2 + i\epsilon} e^{-ik(x-y)}. \tag{10}$$

Compared to the Klein-Gordon case $-g^{\mu\nu}$ gives opposite sign for $\mu = \nu = 0$, and same sign otherwise.

d) So far we have not used the Lorenz condition, which we promised to. But there is a difficulty. Argue that the Lorenz condition $\partial_\mu A^\mu = 0$ is too strong for us, i.e. that it is incompatible with (6).

> **Hint:** Take the divergence with respect to the x coordinate and compare the two sides of the equations if you set $\partial_\mu A^\mu = 0$.

Since we want (6) to be valid, we must relax the Lorenz condition. The way to do this is to express $A^\mu = A^{+\mu} + A^{-\mu}$ (in terms of its positive-frequency and negative-frequency parts), and require that

$$\partial_\mu A^{+\mu} |\psi\rangle = 0 \tag{11}$$

for *physical states* $|\psi\rangle$. This is the so-called Gupta-Bleuler condition. This ensures that the matrix element between physical states, $\langle\psi'| \partial_\mu A^\mu |\psi\rangle = 0$. In other words, the Gupta-Bleuler condition defines a set of physical states, with the property that matrix elements of $\partial_\mu A^\mu$ with these states are zero.

We can write

$$A^{\mu+}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2|\mathbf{k}|}} \sum_{r=0}^{3} \varepsilon_r^\mu(\mathbf{k}) a_{\mathbf{k}}^r e^{-ikx}, \tag{12}$$

which is the usual expression from the massless Klein-Gordon case, however in vectorial form. The unit vectors $\varepsilon_r^\mu(\mathbf{k})$ are chosen such that for each $\mathbf{k}$, $\varepsilon_1(\mathbf{k})$ and $\varepsilon_2(\mathbf{k})$ are perpendicular to $\mathbf{k}$, while $\varepsilon_3(\mathbf{k}) = \mathbf{k}/|\mathbf{k}|$. The final unit vector $\varepsilon_0(\mathbf{k})$ points in the time-like direction.

We obtain the Hamilton density from the Lagrangian in the usual way:

$$\mathcal{H} = \pi_\mu \dot{A}^\mu - \mathcal{L} = -\dot{A}_\mu \dot{A}^\mu + \frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) = -\frac{1}{2}\left(\dot{A}_\mu \dot{A}^\mu + (\partial_i A_\mu)(\partial_i A^\mu)\right). \tag{13}$$

This is a sum over μ; for $\mu = 1, 2, 3$ we obtain the same result as for scalar, massless Klein-Gordon fields, while for $\mu = 0$ we get the opposite sign. Thus, by following the derivation in the Klein-Gordon case, we obtain ((2.31) in P&S, without the infinite vacuum contribution):

$$H = \int \frac{d^3k}{(2\pi)^3} |\mathbf{k}| \left(-a_{\mathbf{k}}^{0\dagger} a_{\mathbf{k}}^0 + a_{\mathbf{k}}^{1\dagger} a_{\mathbf{k}}^1 + a_{\mathbf{k}}^{2\dagger} a_{\mathbf{k}}^2 + a_{\mathbf{k}}^{3\dagger} a_{\mathbf{k}}^3\right). \tag{14}$$

Furthermore, since the canonical momenta are $-\dot{A}^0$ and $\dot{A}^i$, we find the commutator relations

$$[a_{\mathbf{k}}^0, a_{\mathbf{k}}^{0\dagger}] = -1, \quad [a_{\mathbf{k}}^i, a_{\mathbf{k}}^{j\dagger}] = \delta^{ij}, \tag{15}$$

by comparing to the Klein-Gordon case.

e) Show that the Gupta-Bleuler condition requires $\left(a^3_{\mathbf{k}} - a^0_{\mathbf{k}}\right)|\psi\rangle = 0$ for physical states.

> **Hint:** Write $\partial_\mu A^{+\mu} = \partial_i A^{+i} + \partial_0 A^{+0}$, and take the derivative into the integral in (12). Then the first divergence term $\partial_i A^{+i}$ leads to scalar product with $i\mathbf{k}$.

f) Show that

$$\langle\psi| H |\psi\rangle = \langle\psi| \int \frac{d^3k}{(2\pi)^3}|\mathbf{k}|\left(a^{1\dagger}_{\mathbf{k}} a^1_{\mathbf{k}} + a^{2\dagger}_{\mathbf{k}} a^2_{\mathbf{k}}\right)|\psi\rangle \tag{16}$$

for physical states $|\psi\rangle$. Similar conclusions can be made for other, gauge invariant observables. In other words, for free fields *only the transverse photons matters.*

> **Hint:** Make use of the result in the previous question, and its adjungate.

When the electromagnetic field interacts with charges, the situation is different; then longitudinal and scalar photons are present as virtual particles in the process. One may ask how this conclusion can be reconciled with the Coulomb gauge point of view, in which only transverse photons are included. The answer is that the degrees of freedom associated with the scalar photons then are taken care of by the charge density through a Coulomb potential interaction term.

g) The operators $a^0_{\mathbf{k}}$ associated with scalar photons lead to serious problems if they are used in the usual way. If we insist on the commutator relation $[a^0_{\mathbf{k}}, a^{0\dagger}_{\mathbf{k}}] = -1$ and that there exists a state $|0\rangle$ such that $a^0_{\mathbf{k}}|0\rangle = 0$, show that $\langle 0| a^0_{\mathbf{k}} a^{0\dagger}_{\mathbf{k}} |0\rangle = -1$. Since inner products of a vector with itself is non-negative per definition, our starting point must have been wrong.

More details in e.g. Tong ch. 6 or Mandl & Shaw ch. 5. For a more complete treatment of the Gupta-Bleuler formalism using an indefinite metric, see e.g. Cohen-Tannoudji, Dupont-Roc and Grynberg, "Photons and Atoms: Introduction to Quantum Electrodynamics", ch. V.

Problem 2 Identical particles (J. Skaar)

In this exercise, we will consider identical particles, as seen from quantum field theory. Before you start, see the interpretation of the state $\phi(x)|0\rangle$ p. 24 in P&S, and also recall problem set 3, 1e).

a) Consider a two-boson state in the form

$$|\psi\rangle = a^\dagger_1 a^\dagger_2 |0\rangle\,, \tag{17}$$

where

$$a^\dagger_i = \int \frac{d^3p}{(2\pi)^3}\psi_i(\mathbf{p})a^\dagger_{\mathbf{p}},$$

We have seen how the spectra $\psi_1(\mathbf{p})$ and $\psi_1(\mathbf{p})$ can locate the bosons around locations $\mathbf{x}_1$ and $\mathbf{x}_2$; let's assume this is the case. In nonrelatistic quantum mechanics, the state (17) can be written in the form of a wave function $\Psi(\mathbf{x}_1, \mathbf{x}_2)$. Argue from (17) that $\Psi(\mathbf{x}_2, \mathbf{x}_1) = \Psi(\mathbf{x}_1, \mathbf{x}_2)$.

Hint: This exercise is more complicated than it may appear. You can't just say that since $[a_1^\dagger, a_2^\dagger] = 0$, we obtain $\Psi(\mathbf{x}_2, \mathbf{x}_1) = \Psi(\mathbf{x}_1, \mathbf{x}_2)$. You need a precise definition of $\Psi(\mathbf{x}_1, \mathbf{x}_2)$, which is used as a starting point.

Solution: The wave function is given by

$$\Psi(\mathbf{x}_1, \mathbf{x}_2) = \langle 0| \phi(x_2)\phi(x_1) a_1^\dagger a_2^\dagger |0\rangle \,, \tag{18}$$

where the times x_1^0 and x_2^0 are equal. This gives

$$\Psi(\mathbf{x}_1, \mathbf{x}_2) = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-ipx_1 - iqx_2}}{\sqrt{2E_p 2E_q}} \langle 0| a_{\mathbf{q}} a_{\mathbf{p}} a_1^\dagger a_2^\dagger |0\rangle$$

We have $[a_{\mathbf{p}}, a_1^\dagger] = \psi_1(\mathbf{p})$, and therefore

$$\begin{aligned}\Psi(\mathbf{x}_1, \mathbf{x}_2) &= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-ipx_1 - iqx_2}}{\sqrt{2E_p 2E_q}} [\psi_1(\mathbf{p})\psi_2(\mathbf{q}) + \psi_1(\mathbf{q})\psi_2(\mathbf{p})] \\ &= \Psi_1(x_1)\Psi_2(x_2) + \Psi_1(x_2)\Psi_2(x_1),\end{aligned}$$

with

$$\Psi_i(x) = \int \frac{d^3p}{(2\pi)^3} \frac{\psi_i(\mathbf{p})}{\sqrt{2E_{\mathbf{p}}}} e^{-ipx}.$$

Clearly we have $\Psi(\mathbf{x}_1, \mathbf{x}_2) = \Psi(\mathbf{x}_2, \mathbf{x}_1)$.

b) If the ladder operators $a_1^\dagger$ and $a_2^\dagger$ (or $a_{\mathbf{p}}^\dagger$ and $a_{\mathbf{q}}^\dagger$) anticommuted rather than commuted, how would the conclusion change?

Solution: By going through the derivation, we obtain the fermionic property $\Psi(\mathbf{x}_1, \mathbf{x}_2) = -\Psi(\mathbf{x}_2, \mathbf{x}_1)$.

c) Consider now the case with two distinguishable bosons. Then not only $a_{\mathbf{p}}^\dagger$ and $b_{\mathbf{q}}^\dagger$ commute, but also $a_{\mathbf{p}}$ and $b_{\mathbf{q}}^\dagger$ commute. Here $a_{\mathbf{p}}^\dagger$ and $b_{\mathbf{q}}^\dagger$ create the two different particles. Show that the wave function can be expressed

$$\Psi(x_1, x_2) = \Psi_1(x_1)\Psi_2(x_2), \tag{19}$$

for some wave functions $\Psi_1(x_1)$ and $\Psi_2(x_2)$.