

University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set Trial exam

T. Klungland and J. Skaar

Problem 1 Inhomogeneous magnetization

In this problem we consider the free, quantized Dirac theory, and show how the theory leads to an effective current density due to an inhomogeneous spin density.

The one-particle fermionic state with spin label s and momentum $\mathbf{p}$ is given by

$$|\mathbf{p}, s\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^{s\dagger} |0\rangle, \quad E_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2. \quad (1)$$

Our fermion will be expressed as

$$|\xi, s\rangle = a_{\xi}^{s\dagger} |0\rangle, \quad a_{\xi}^{s\dagger} = \int \frac{d^3p}{(2\pi)^3} \xi(\mathbf{p}) a_{\mathbf{p}}^{s\dagger}, \quad (2)$$

where $a_{\xi}^{s\dagger}$ should be viewed as a wavepacket creation operator. The wavepacket spectrum $\xi(\mathbf{p})$ is normalized by

$$\int \frac{d^3p}{(2\pi)^3} |\xi(\mathbf{p})|^2 = 1.$$

a) Calculate $\{a_{\xi}^s, a_{\xi}^{s\dagger}\}$ and $\langle \xi, s | \xi, s \rangle = \langle 0 | a_{\xi}^s a_{\xi}^{s\dagger} | 0 \rangle$.

b) Show that the expectation value of the observable $\bar{\psi}\psi$ in the state (2) is

$$\langle \bar{\psi}\psi \rangle = \bar{\Psi}(x) \Psi(x), \quad (3)$$

up to an additive (infinite) constant. Here we have defined

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{\xi(\mathbf{p})}{\sqrt{2E_{\mathbf{p}}}} u^s(\mathbf{p}) e^{-ipx}.$$

We now consider the nonrelativistic limit. Then the wave function $\Psi(x)$ satisfies the Schrödinger equation, as can be seen by an identical argument as that used for the scalar bosonic field. Also, since there are only two linearly independent $u^s(p)$'s (positive-frequency solutions to the Dirac equation), they span a space of two-dimensional spinors, just as in nonrelativistic quantum mechanics.

c) Show that in the nonrelativistic limit, $\langle \bar{\psi}\psi \rangle$ and the charge density $\langle \bar{\psi}\gamma^0\psi \rangle$ are independent of the spin state.

d) Still in the nonrelativistic limit, show that the current density $\langle \bar{\psi}\gamma^i\psi \rangle$ is generally dependent on the spin state, but if the state contains a sufficiently narrow band of $\mathbf{p}$'s, the dependence is negligible.

e) Prove that the spin dependent part of the current density $\langle \bar{\psi} \boldsymbol{\gamma} \psi \rangle$ can be written $\nabla \times \mathbf{M}$, where $\mathbf{M} = \Psi^\dagger \boldsymbol{\sigma} \Psi = g \Psi^\dagger \frac{\boldsymbol{\sigma}}{2} \Psi$, and Ψ is defined as before, but with the Dirac-spinor $u^s(\mathbf{p})$ replaced by the two-component spinor ξ^s . The parameter $g = 2$ is the Landé g -factor.

Give a physical interpretation.

Problem 2 Electron-muon scattering

Here we consider the process $e^-(p_1)\mu^-(p_2) \rightarrow e^-(p_3)\mu^-(p_4)$, to leading order.

a) Argue why, even in the low-energy limit, it is a reasonable approximation to treat the electron as massless, at this order in perturbation theory.

b) Draw the leading-order ($\mathcal{O}(e^2)$) Feynman diagram for this process, and write down its amplitude.

c) Calculate the spin-averaged, unpolarized differential cross-section $\frac{d\sigma}{d\Omega}$ for this process, as a function of the muon mass m_μ , in the center-of-mass frame. You do not need to carry out the kinematics to find explicit expressions for the energies of the particles; it suffices to parametrize the momenta as

$$p_1 = (k, 0, 0, k), \quad (4)$$

$$p_2 = (E, 0, 0, -k), \quad (5)$$

$$p_3 = (k, k \sin \theta, 0, k \cos \theta), \quad (6)$$

$$p_4 = (E, -k \sin \theta, 0, -k \cos \theta), \quad (7)$$

where $k^2 + m_\mu^2 = E^2$ and $E + k \equiv E_{\text{cm}}$.

d) The expression for the squared matrix element could have been obtained directly, without the from-scratch evaluations of the previous problems, from the corresponding expression for $e^+e^- \rightarrow \mu^+\mu^-$ (which you have worked on in the lectures). Explain how this is done.

e) Draw one $\mathcal{O}(e^4)$ diagram that contributes to the S matrix; explain the types of divergence that can appear in this diagram and how these are handled.

f) This process is very similar to another process called Møller scattering: $e^-e^- \rightarrow e^-e^-$. For Møller scattering there is an additional leading-order diagram, however; draw this diagram and write down its amplitude, including its sign relative to the one from part a).

Problem 3 Electron self-energy

Let $-i\Sigma(\not{p})$ be the sum of all one-particle irreducible diagram insertions into the fermion propagator, which we assume known. (To get a meaningful Feynman diagram, the sum $-i\Sigma(\not{p})$ must be accompanied by a fermion

propagator on each side, which are excluded from $-i\Sigma(\not{p})$.) The fermion propagator (with the bare mass m_0) is written as

$$\frac{i(\not{p} + m_0)}{p^2 - m_0^2} \equiv \frac{i}{\not{p} - m_0}. \quad (8)$$

The sum of all two-point diagrams contains a propagator

$$\frac{iZ}{\not{p} - m}, \quad (9)$$

where Z is a constant, and m is different from m_0 . Find the relation between m and m_0 .

Formulas

Lagrangian field theory:

$$\begin{aligned} S &= \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \\ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} &= 0 \\ \pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ H &= \int d^3x (\pi \dot{\phi} - \mathcal{L}) \end{aligned}$$

Noether:

$$\begin{aligned} \phi_r &\mapsto \phi_r + \alpha \Delta \phi_r \\ \mathcal{L} &\mapsto \mathcal{L} + \alpha \partial_\mu J^\mu \\ j^\mu &= \sum_r \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} \Delta \phi_r - J^\mu \end{aligned}$$

Lorentz transformations, boost along the z -axis:

$$\begin{aligned} t' &= \gamma(t + \beta z) \\ z' &= \gamma(z + \beta t) \end{aligned}$$

Klein-Gordon field:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right)$$

Dirac field:

$$\begin{aligned} \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_r \left(a_{\mathbf{p}}^r u^r(\mathbf{p}) e^{-ipx} + b_{\mathbf{p}}^{r\dagger} v^r(\mathbf{p}) e^{ipx} \right) \\ u^s(p) &= \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix}, \quad v^s(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix} \\ \bar{u}^r(p) u^r(p) &= -\bar{v}^r(p) v^r(p) = 2m \delta^{rs} \\ \sum_{s=1}^2 u^s(p) \bar{u}^s(p) &= \not{p} + m, \quad \sum_{s=1}^2 v^s(p) \bar{v}^s(p) = \not{p} - m \end{aligned}$$

Chiral representation:

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}.$$

Definitions:

$$f \overset{\leftrightarrow}{\partial}_0 g = f \partial_0 g - (\partial_0 f) g$$

Identities:

$$\begin{aligned} \sigma^i \sigma^j &= \delta^{ij} + i\epsilon^{ijk} \sigma^k \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\ \gamma^\mu \gamma_\mu &= 4 \\ \gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma_\mu &= 4g^{\nu\alpha} \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma_\mu &= -2\gamma^\rho \gamma^\alpha \gamma^\nu \\ \text{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu} \\ \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\rho} g^{\nu\sigma} + 4g^{\mu\sigma} g^{\nu\rho} \\ \text{Tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] &= 0, \text{ for } n \text{ odd} \end{aligned}$$

Cross-sections and decay rates:

$$\begin{aligned} d\sigma &= \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_A, p_B \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A + p_B - \sum p_f) \\ d\Gamma &= \frac{1}{2m_A} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_A = (m_A, \mathbf{0}) \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A - \sum p_f) \end{aligned}$$

Two-particle phase space:

$$\int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2) = \int d\Omega_{\text{cm}} \frac{1}{16\pi^2} \frac{|\mathbf{p}_1|^2}{E_{\text{cm}}},$$

$E_{\text{cm}} = E_1 + E_2$ in the center-of-mass frame; Ω_{cm} = solid angle for particle 1. For decay of a single particle, set $p_B = 0$.

Lagrangians and Feynman rules

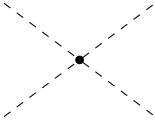
The following rules are used to obtain the amplitude $i\mathcal{M}$, and given in momentum space. Dots represent vertices in all of the diagrams listed below. For consistency between ϕ^4 theory and Yukawa theory, scalar particles are represented by dashed lines, and their masses by M , in both cases.

Common for all theories:

- Impose momentum conservation at each vertex
- Integrate over each undetermined momentum, $\int \frac{d^4 k}{(2\pi)^4}$

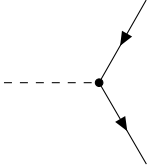
ϕ^4 theory

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

- Propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$
- External line: $\xrightarrow{p} \bullet = 1$
- Vertex:  $= -i\lambda$
- Divide by the symmetry factor

Yukawa theory:

$$\mathcal{L}_Y = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2 \phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\psi\phi$$

- Scalar propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$
- Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$
- External scalar: $\xrightarrow{p} \bullet = 1$
- Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$
- Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$
- Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$
- Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$
- Vertex:  $= -ig$
- Work out the overall sign of the diagram

QED:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad e = -|e|$$

- Photon propagator: $\mu \xrightarrow{p} \nu = \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}$

- Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon};$
- Incoming photon: $\mu \xrightarrow{p} \bullet = \epsilon_\mu(p)$
- Outgoing photon: $\bullet \xrightarrow{p} \mu = \epsilon_\mu^*(p)$
- Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$
- Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$
- Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$
- Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$
- Vertex: $\mu \xrightarrow{\quad} \bullet \begin{array}{l} \nearrow \\ \searrow \end{array} = -ie\gamma^\mu$
- Work out the overall sign of the diagram