

University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Exam 20.12.2022

Problem 1 Lorentz invariance and causality

- a) Which of the following quantities are four-vectors? Position vector in 3D space $\mathbf{x}$, four-momentum p^μ , Dirac field ψ^α , and $\bar{\psi}\gamma^\mu\psi$.
- b) Let $x = (t, \mathbf{x})$ be a spacelike four-vector. Prove that there exists a boost $x \mapsto x' = (t', \mathbf{x}')$ such that $t' = 0$.
- c) Consider the free Klein-Gordon theory. Prove that a measurement of $\phi(x)$ cannot affect a measurement of $\phi(y)$ if $x - y$ is spacelike.

Problem 2 Two interacting fields

Consider two interacting, real, scalar fields ϕ_1 and ϕ_2 , described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) - \frac{1}{2}m_2^2\phi_2^2 - \lambda\phi_1^2\phi_2^2, \quad (1)$$

where λ is a constant.

- a) Derive the equations of motion for the theory (1). Do so by directly using the principle of least action (by varying the fields), i.e., not by using the Euler-Lagrange equations. Give an explanation of why the boundary terms do not contribute in your result, and discuss the relation to the Klein-Gordon-equation(s) you would expect for non-interacting fields.
- b) Determine the Feynman rules of the theory. Ignore combinatorics / symmetry factors. You may take the usual Feynman rules for ϕ^4 theory for granted. Also recall the formula

$$\langle \mathbf{p}'\mathbf{k}' | S | \mathbf{p}\mathbf{k} \rangle = {}_0\langle \mathbf{p}'\mathbf{k}' | T e^{-i \int dt H_I(t)} | \mathbf{p}\mathbf{k} \rangle_{0 \text{ connected, amputated}}, \quad (2)$$

where “0” stands for free eigenstates, $\mathbf{p}$, $\mathbf{k}$, $\mathbf{p}'$, and $\mathbf{k}'$ are the momenta of any of the two particle types, and $H_I(t)$ is the interaction Hamiltonian written in the interaction picture.

Give an example of an $\mathcal{O}(\lambda^2)$ process, in terms of a Feynman diagram and the corresponding expression for $i\mathcal{M}$. (Leave any momentum integrals undetermined.)

Problem 3 Scalar decay in Yukawa theory

This problem explores the decay of the scalar to a fermion-antifermion pair in Yukawa theory (with Lagrangian and Feynman rules listed in the formula collection). This is actually a very important process in the Standard Model, as the (scalar) Higgs boson interaction with fermions is described by exactly such a theory, and in particular $h \rightarrow b\bar{b}$ (where h is the Higgs boson, and b is the second-heaviest quark) is the dominant decay channel for the Higgs boson. Let M be the mass of the scalar, and m the mass of each of the fermions.

- a) Draw the leading-order (first order in the coupling parameter g) Feynman diagram for this decay process, and write down its amplitude.
- b) Calculate the leading-order decay rate Γ of the scalar, assuming that we do not measure the spin of the final-state fermions. (See formula collection.)
- c) Argue that the analogous QED process, $\gamma \rightarrow e^+e^-$, is not possible.

Problem 4 LSZ reduction

In this problem we consider a self-interacting, scalar theory with field $\phi(x)$. Define operators

$$a_{\mathbf{p}}^{\psi}(t) = i \int d^3x \Psi_{\mathbf{p}}^*(x) \overleftrightarrow{\partial}_0 \phi(x), \quad (3a)$$

$$a_{\mathbf{p}}^{\psi\dagger}(t) = -i \int d^3x \Psi_{\mathbf{p}}(x) \overleftrightarrow{\partial}_0 \phi(x), \quad (3b)$$

where

$$\Psi_{\mathbf{p}}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{\psi(\mathbf{k} - \mathbf{p})}{\sqrt{2E_{\mathbf{k}}}} e^{-ikx}, \quad k^0 = E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}, \quad (4)$$

with the single particle mass m in the interacting theory. Here $\psi(\mathbf{k})$ is a spectrum centered about $\mathbf{k} = 0$, e.g., a gaussian.

- a) Taking ϕ^4 theory as an example, use

$$\begin{aligned} & \langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle \\ &= \frac{\langle 0 | T \{ \phi_I(x_1) \cdots \phi_I(x_n) e^{-i \int dt H_I(t)} \} | 0 \rangle}{\langle 0 | T \{ e^{-i \int dt H_I(t)} \} | 0 \rangle}, \quad n \geq 1, \end{aligned} \quad (5)$$

to argue that

$$\langle \Omega | \phi(x) | \Omega \rangle = 0. \quad (6)$$

Here T denotes time ordering, $H_I(t)$ is the interaction Hamiltonian written in the interaction picture, and $\phi_I(x)$ is the field in the interaction picture.

- b) Let $|\lambda_{\mathbf{p}}\rangle$ be the eigenstates of the total Hamiltonian H and the momentum operator $\mathbf{P}$, with eigenvalues $p^0 = E_{\mathbf{p}}^{\lambda} \equiv \sqrt{\mathbf{p}^2 + m_{\lambda}^2}$ and $\mathbf{p}$, respectively. Here $m_{\lambda} > 0$ is the energy eigenvalue in the rest frame. From Poincaré symmetry we have

$$\langle \Omega | \phi(x) | \lambda_{\mathbf{p}} \rangle = \langle \Omega | \phi(0) | \lambda_0 \rangle e^{-ipx}, \quad p^0 = E_{\mathbf{p}}^{\lambda}. \quad (7)$$

Prove that

$$\langle \Omega | a_{\mathbf{p}}^{\psi}(t) | \Omega \rangle = 0. \quad (8)$$

Also calculate the matrix element

$$\langle \lambda_{\mathbf{p}} | a_{\mathbf{q}}^{\psi}(\pm\infty) | \Omega \rangle \quad (9)$$

and argue that it vanishes if $\langle \lambda_{\mathbf{p}} |$ is replaced by a (possibly narrow-banded) wavepacket (superposition of $\langle \lambda_{\mathbf{p}} |$ for a narrow band of $\mathbf{p}$'s).

From this argue that

$$a_{\mathbf{q}}^{\psi}(\pm\infty) | \Omega \rangle = 0. \quad (10)$$

c) Derive an expression for $\langle \Omega | \phi(y) a_{\mathbf{k}}^{\psi\dagger}(-\infty) | \Omega \rangle$ in terms of the two-point correlation function $\langle \Omega | T \phi(y) \phi(x) | \Omega \rangle$. The result should be as simple as possible. It should also be written in four-dimensional (manifestly covariant) form, except for the presence of $\Psi_{\mathbf{k}}(x)$. *Hint:* Use (10) to insert a term with another creation operator.

d) Why is the LSZ reduction in the previous question (generalized to more than two fields) useful?

Formulas

Lagrangian field theory:

$$\begin{aligned} S &= \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \\ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= 0 \\ \pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ H &= \int d^3x (\pi \dot{\phi} - \mathcal{L}) \end{aligned}$$

Noether:

$$\begin{aligned} \phi_r &\mapsto \phi_r + \alpha \Delta \phi_r \\ \mathcal{L} &\mapsto \mathcal{L} + \alpha \partial_\mu J^\mu \\ j^\mu &= \sum_r \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \Delta \phi_r - J^\mu \end{aligned}$$

Lorentz transformations, boost along the z -axis:

$$\begin{aligned} t' &= \gamma(t + \beta z) \\ z' &= \gamma(z + \beta t) \end{aligned}$$

Klein-Gordon field:

$$\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right) \\ [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned}$$

Dirac field:

$$\begin{aligned} \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_r \left(a_{\mathbf{p}}^r u^r(\mathbf{p}) e^{-ipx} + b_{\mathbf{p}}^{r\dagger} v^r(\mathbf{p}) e^{ipx} \right) \\ u^s(p) &= \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix}, \quad v^s(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix} \\ \bar{u}^r(p) u^s(p) &= -\bar{v}^r(p) v^s(p) = 2m \delta^{rs} \\ \sum_{s=1}^2 u^s(p) \bar{u}^s(p) &= \not{p} + m, \quad \sum_{s=1}^2 v^s(p) \bar{v}^s(p) = \not{p} - m \end{aligned}$$

Chiral representation:

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}.$$

Definitions:

$$f \overset{\leftrightarrow}{\partial}_0 g = f \partial_0 g - (\partial_0 f) g$$

Identities:

$$\begin{aligned}\sigma^i \sigma^j &= \delta^{ij} + i \epsilon^{ijk} \sigma^k \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\ \gamma^\mu \gamma_\mu &= 4 \\ \gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma_\mu &= 4g^{\nu\alpha} \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma_\mu &= -2\gamma^\rho \gamma^\alpha \gamma^\nu \\ \text{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu} \\ \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\rho} g^{\nu\sigma} + 4g^{\mu\sigma} g^{\nu\rho} \\ \text{Tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] &= 0, \text{ for } n \text{ odd}\end{aligned}$$

Cross-sections and decay rates:

$$\begin{aligned}d\sigma &= \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_A, p_B \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A + p_B - \sum p_f) \\ d\Gamma &= \frac{1}{2m_A} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_A = (m_A, \mathbf{0}) \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A - \sum p_f)\end{aligned}$$

Two-particle phase space:

$$\int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2) = \int d\Omega_{\text{cm}} \frac{1}{16\pi^2} \frac{|\mathbf{p}_1|}{E_{\text{cm}}},$$

$E_{\text{cm}} = E_1 + E_2$ in the center-of-mass frame; Ω_{cm} = solid angle for particle 1. For decay of a single particle, set $p_B = 0$.

Lagrangians and Feynman rules

The following rules are used to obtain the amplitude $i\mathcal{M}$, and given in momentum space. Dots represent vertices in all of the diagrams listed below. For consistency between ϕ^4 theory and Yukawa theory, scalar particles are represented by dashed lines, and their masses by M , in both cases.

Common for all theories:

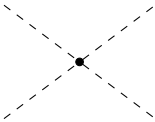
- Impose momentum conservation at each vertex
- Integrate over each undetermined momentum, $\int \frac{d^4 k}{(2\pi)^4}$

ϕ^4 theory

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

• Propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

• External line: $\xrightarrow{p} \bullet = 1$

• Vertex:  $= -i\lambda$

- Divide by the symmetry factor

Yukawa theory:

$$\mathcal{L}_Y = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2 \phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\psi\phi$$

• Scalar propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

• Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

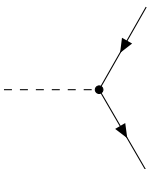
• External scalar: $\xrightarrow{p} \bullet = 1$

• Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$

• Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$

• Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$

• Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$

• Vertex:  $= -ig$

- Work out the overall sign of the diagram

QED:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad e = -|e|$$

• Photon propagator: $\mu \xrightarrow{p} \nu = \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}$

- Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$;
- Incoming photon: $\mu \xrightarrow{p} \bullet = \epsilon_\mu(p)$
- Outgoing photon: $\bullet \xrightarrow{p} \mu = \epsilon_\mu^*(p)$
- Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$
- Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$
- Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$
- Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$
- Vertex: $\mu \xrightarrow{\quad} \bullet \begin{array}{l} \nearrow \\ \searrow \end{array} = -ie\gamma^\mu$
- Work out the overall sign of the diagram