

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: FYS4170 — Relativistic Quantum Field Theory

Day of examination: 19. december 2025

Examination hours: 09.00 – 13.00

This problem set consists of 7 pages.

Appendices: Formula sheets are included.

Permitted aids: No examination support material is allowed.

Make sure that your copy of this examination paper is complete before answering.

Problem 1

a) What is a four-vector?

b) Wick's theorem for two fermion fields is

$$T\psi(x)\bar{\psi}(y) = N\psi(x)\bar{\psi}(y) + \overline{\psi(x)\psi(y)}. \quad (1)$$

If (1) is taken as the definition of the contraction $\overline{\psi(x)\psi(y)}$, prove the two equalities

$$\begin{aligned} \overline{\psi(x)\psi(y)} &= \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle \\ &= \begin{cases} \{\psi^+(x), \bar{\psi}^-(y)\}, & x^0 > y^0, \\ -\{\bar{\psi}^+(y), \psi^-(x)\}, & x^0 < y^0, \end{cases} \end{aligned} \quad (2)$$

where $\psi^+(x)$ and $\psi^-(x)$ are the positive and negative frequency parts of $\psi(x)$.

c) What is renormalization in quantum field theory? What does it mean that a theory is renormalizable? (About 5-10 sentences.)

Problem 2

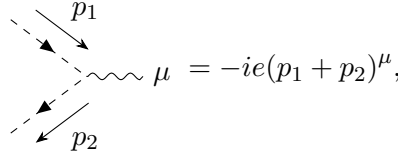
In this problem we will investigate a version of QED that contains both a charged fermion and scalar boson, both of which are coupled to the photon field. Assuming all particles to be *massless*, the Lagrangian of the theory is given by

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) + \bar{\psi}(i\not{D})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (3)$$

where $D_\mu = \partial_\mu + ieA_\mu$.

a) Derive the Euler-Lagrange equations for $\phi^\dagger$ and $\bar{\psi}$ (which become equations of motion for ϕ and ψ). Write the result with the covariant derivative D_μ .

b) This theory has three interaction vertices. One is given by



$$\mu = -ie(p_1 + p_2)^\mu, \quad (4)$$

where the dashed lines represent charged scalars. Draw the two other vertices, and give their order in e (you do not need to derive the full Feynman rules).

c) Draw the leading-order Feynman diagram for the process $\bar{\psi}\psi \mapsto \phi^\dagger\phi$ (i.e., a fermion-antifermion pair $\mapsto$ scalar boson particle-antiparticle pair). Show that the differential cross-section in the center-of-mass frame $\frac{d\sigma}{d\Omega_{\text{cm}}}$ for the process, to lowest order in e , is given by

$$\frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{\alpha^2}{8E_{\text{cm}}^2} \sin^2 \theta, \quad (5)$$

where $\alpha = e^2/(4\pi)$, E_{cm} is the center-of-mass energy, and θ is the angle between the incoming and outgoing particle directions in the center-of-mass frame.

Problem 3

In this problem we consider a free, scalar, hermittian quantum field $\phi(x)$. The field satisfies the Klein-Gordon equation with mass m , and can be written in the usual form (see formula sheet).

a) Which of the following quantities must vanish due to causality?

1. the correlation function $\langle 0|\phi(x)\phi(y)|0\rangle$ for any $x - y$;
2. the correlation function $\langle 0|\phi(x)\phi(y)|0\rangle$ for spacelike $x - y$;
3. the correlation function $\langle 0|\phi(x)\phi(y)|0\rangle$ for timelike $x - y$;
4. the commutator $[\phi(x), \phi(y)]$ for any $x - y$;
5. the commutator $[\phi(x), \phi(y)]$ for spacelike $x - y$;
6. the commutator $[\phi(x), \phi(y)]$ for timelike $x - y$;
7. the expectation value of $\phi(x)\phi(x)$ in a single particle state for all x ;
8. the expectation value of $\phi(x)\phi(x)$ in a single particle state for $\mathbf{x} \rightarrow \infty$ when the particle is located in the origin.

b) Show that the correlation function $D(x - y) \equiv \langle 0 | \phi(x) \phi(y) | 0 \rangle$ can be written in the form

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2E_{\mathbf{k}}} e^{-ik \cdot (x-y)}. \quad (6)$$

From now on, we consider the field at $t = 0$, and denote it $\phi(\mathbf{x})$. The field $\phi(\mathbf{x})$ itself is not a proper observable, since it is a so-called operator-valued distribution. This is seen e.g. from the commutator relation, involving a Dirac delta distribution. To obtain an observable ϕ_σ , we must therefore integrate (smear) the fields with a test function $f_\sigma(\mathbf{x})$:

$$\phi_\sigma \equiv \int d^3 x f_\sigma(\mathbf{x}) \phi(\mathbf{x}). \quad (7)$$

For ϕ_σ to represent the field close to $\mathbf{x} = 0$, this test function must be nonzero only in a small neighborhood of $\mathbf{x} = 0$. We let $f_\sigma(\mathbf{x})$ be 3D gaussian distribution

$$f_\sigma(\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}} e^{-\frac{|\mathbf{x}|^2}{2\sigma^2}}. \quad (8)$$

This function is normalized $\int d^3 x f_\sigma(\mathbf{x}) = 1$ and has Fourier transform

$$F_\sigma(\mathbf{k}) \equiv \int d^3 x f_\sigma(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} = e^{-\frac{\sigma^2 |\mathbf{k}|^2}{2}}. \quad (9)$$

c) Calculate the expectation value of the smeared field in the vacuum state, $\langle 0 | \phi_\sigma | 0 \rangle$.

d) Find an expression for $\langle 0 | \phi_\sigma^2 | 0 \rangle$, and show that for the case $\sigma \gg m^{-1}$ it is roughly

$$\langle 0 | \phi_\sigma^2 | 0 \rangle = \frac{1}{16\pi^{\frac{3}{2}} m \sigma^3}. \quad (10)$$

Given: The Γ function is defined by

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t}, \quad (11)$$

with some values $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$, and recursive formula $\Gamma(z+1) = \Gamma(z)z$.

More generally, one can show (you don't have to do this):

$$\langle 0 | \phi_\sigma^2 | 0 \rangle = \begin{cases} \frac{1}{8\pi^2 \sigma^2}, & \sigma \ll m^{-1}, \\ \frac{1}{16\pi^{\frac{3}{2}} m \sigma^3}, & \sigma \gg m^{-1}. \end{cases} \quad (12)$$

e) What is the physical significance of the expression $\langle 0 | \phi_\sigma^2 | 0 \rangle$? Explain what the result (12) means physically.

Formulas

Lagrangian field theory:

$$\begin{aligned}
 S &= \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \\
 \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} &= 0 \\
 \pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
 H &= \int d^3x (\pi \dot{\phi} - \mathcal{L})
 \end{aligned}$$

Noether:

$$\begin{aligned}
 \phi_r &\mapsto \phi_r + \alpha \Delta \phi_r \\
 \mathcal{L} &\mapsto \mathcal{L} + \alpha \partial_\mu J^\mu \\
 j^\mu &= \sum_r \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} \Delta \phi_r - J^\mu
 \end{aligned}$$

Lorentz transformations, boost along the z -axis:

$$\begin{aligned}
 t' &= \gamma(t + \beta z) \\
 z' &= \gamma(z + \beta t)
 \end{aligned}$$

Klein-Gordon field:

$$\begin{aligned}
 \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right) \\
 (\partial^2 + m^2)\phi &= 0 \\
 [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}).
 \end{aligned}$$

Dirac field:

$$\begin{aligned}
 \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_r \left(a_{\mathbf{p}}^r u^r(\mathbf{p}) e^{-ipx} + b_{\mathbf{p}}^{r\dagger} v^r(\mathbf{p}) e^{ipx} \right) \\
 (i\not{\partial} - m)\psi &= 0 \\
 \{a_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}\} &= \{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs} \\
 u^s(p) &= \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix}, \quad v^s(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix} \\
 \bar{u}^r(p) u^s(p) &= -\bar{v}^r(p) v^s(p) = 2m \delta^{rs} \\
 \not{p} u(p) &= m u(p), \quad \bar{u}(p) \not{p} = \bar{u}(p) m, \\
 \not{p} v(p) &= -m v(p), \quad \bar{v}(p) \not{p} = -\bar{v}(p) m \\
 \sum_{s=1}^2 u^s(p) \bar{u}^s(p) &= \not{p} + m, \quad \sum_{s=1}^2 v^s(p) \bar{v}^s(p) = \not{p} - m
 \end{aligned}$$

Chiral representation:

$$\begin{aligned}
 \gamma^0 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}; \\
 \sigma^1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
 \end{aligned}$$

Definitions:

$$\begin{aligned} f \overset{\leftrightarrow}{\partial}_0 g &= f \partial_0 g - (\partial_0 f) g \\ \sigma^\mu &\equiv (1, \sigma^i), \quad \bar{\sigma}^\mu \equiv (1, -\sigma^i) \\ \not{a} &\equiv \gamma^\mu a_\mu \text{ (4-vector } a^\mu) \end{aligned}$$

Identities:

$$\begin{aligned} \sigma^i \sigma^j &= \delta^{ij} + i\epsilon^{ijk} \sigma^k \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\ \gamma^5 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \\ \gamma^\mu \gamma_\mu &= 4 \\ \gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma_\mu &= 4g^{\nu\alpha} \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma_\mu &= -2\gamma^\rho \gamma^\alpha \gamma^\nu \\ \text{tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu} \\ \text{tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\rho} g^{\nu\sigma} + 4g^{\mu\sigma} g^{\nu\rho} \\ \text{tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] &= 0, \text{ for } n \text{ odd} \end{aligned}$$

$(\bar{v}^s(p) \Gamma u^r(k))^\dagger = \bar{u}^r(k) \Gamma' v^s(p)$, Gamma matrices in Γ' reversed compared to that in Γ .

Ward's identity:

Let $i\mathcal{M}_\mu(q)$ be a diagram where a photon line with four-momentum q at vertex μ is removed.

Then $\mathcal{M}_\mu(q)q^\mu = 0$.

Gordon's identity:

$$\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p), \text{ where } \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \text{ and } q^\mu = p'^\mu - p^\mu.$$

Cross-sections and decay rates:

$$\begin{aligned} d\sigma &= \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_A, p_B \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A + p_B - \sum p_f) \\ d\Gamma &= \frac{1}{2m_A} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_A = (m_A, \mathbf{0}) \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A - \sum p_f) \end{aligned}$$

Two-particle phase space:

$$\int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2) = \int d\Omega_{\text{cm}} \frac{1}{16\pi^2} \frac{|\mathbf{p}_1|}{E_{\text{cm}}},$$

$E_{\text{cm}} = E_1 + E_2$ in the center-of-mass frame; Ω_{cm} = solid angle for particle

1. For decay of a single particle, set $p_B = 0$.

Lagrangians and Feynman rules

The following rules are used to obtain the amplitude $i\mathcal{M}$, and given in momentum space. Dots represent vertices in all of the diagrams listed below. For consistency between ϕ^4 theory and Yukawa theory, scalar particles are represented by dashed lines, and their masses by M , in both cases.

Common for all theories:

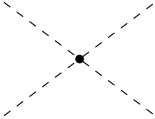
- Impose momentum conservation at each vertex
- Integrate over each undetermined momentum, $\int \frac{d^4k}{(2\pi)^4}$

 ϕ^4 theory

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

• Propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

• External line: $\xrightarrow{p} \bullet = 1$

• Vertex:  $= -i\lambda$

- Divide by the symmetry factor

Yukawa theory:

$$\mathcal{L}_Y = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\psi\phi$$

• Scalar propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

• Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

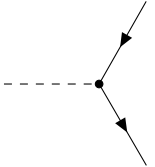
• External scalar: $\xrightarrow{p} \bullet = 1$

• Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$

• Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$

• Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$

• Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$

• Vertex:  $= -ig$

- Work out the overall sign of the diagram
- For closed fermion loops, include a factor (-1) and a trace.

QED:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad e = -|e|$$

- Photon propagator: $\mu \xrightarrow{p} \nu = \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}$
- Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$
- Incoming photon: $\mu \xrightarrow{p} = \epsilon_\mu(p)$
- Outgoing photon: $\bullet \xrightarrow{p} \mu = \epsilon_\mu^*(p)$
- Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$
- Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$
- Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$
- Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$
- Vertex: $\mu \xrightarrow{\quad} \bullet = -ie\gamma^\mu$

- Work out the overall sign of the diagram
- For closed fermion loops, include a factor (-1) and a trace.