

University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Exam 20.12.2022

Problem 1 Lorentz invariance and causality

a) Which of the following quantities are four-vectors? Position vector in 3D space $\mathbf{x}$, four-momentum p^μ , Dirac field ψ^α , and $\bar{\psi}\gamma^\mu\psi$.

b) Let $x = (t, \mathbf{x})$ be a spacelike four-vector. Prove that there exists a boost $x \mapsto x' = (t', \mathbf{x}')$ such that $t' = 0$.

Solution: First rotate spatial coordinates such that $x = (t, 0, 0, z)$. Then apply a boost along the z -direction: $t' = \gamma(t + \beta z)$. We obtain $t' = 0$ provided $\beta = -t/z$. When x is spacelike, we have $t/z < 1$ so $|\beta| < 1$; thus this is a valid boost.

c) Consider the free Klein-Gordon theory. Prove that a measurement of $\phi(x)$ cannot affect a measurement of $\phi(y)$ if $x - y$ is spacelike.

Solution: We have

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \\ &\quad \cdot [a_{\mathbf{p}}e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}, a_{\mathbf{q}}e^{-iqy} + a_{\mathbf{q}}^\dagger e^{iqy}] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (e^{-ip(x-y)} - e^{ip(x-y)}). \end{aligned} \quad (1)$$

The result is Lorentz invariant (recall that $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}}$ is Lorentz invariant). Applying a Lorentz transformation to eliminate the time-like component of $x - y$ (since $x - y$ is spacelike this is possible according to the previous question), we get

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (e^{i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})}). \quad (2)$$

Changing integration variable $\mathbf{p} \rightarrow -\mathbf{p}$ for the last term, and using that $E_{-\mathbf{p}} = E_{\mathbf{p}}$, we find that the two terms exactly cancel.

Since the commutator $[\phi(x), \phi(y)]$ vanishes, a measurement at x cannot affect a measurement at y .

Problem 2 Two interacting fields

Consider two interacting, real, scalar fields ϕ_1 and ϕ_2 , described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) - \frac{1}{2}m_1^2 \phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2}m_2^2 \phi_2^2 - \lambda \phi_1^2 \phi_2^2, \quad (3)$$

where λ is a constant.

a) Derive the equations of motion for the theory (3). Do so by directly using the principle of least action (by varying the fields), i.e., not by using the Euler-Lagrange equations. Give an explanation of why the boundary terms do not contribute in your result, and discuss the relation to the Klein-Gordon-equation(s) you would expect for non-interacting fields.

Solution: We consider the two real scalar fields ϕ_i ($i = 1, 2$) as independent. Under a change of one of them, say $\phi_1 \rightarrow \phi_1 + \delta\phi_1$, the Lagrangian changes as $\mathcal{L} \rightarrow \mathcal{L} + \delta\mathcal{L}$ with

$$\delta\mathcal{L} = (\partial_\mu\phi_1)(\partial^\mu\delta\phi_1) - m_1^2\phi_1\delta\phi_1 - 2\lambda\phi_1\phi_2^2\delta\phi_1,$$

where we used the fact that $\delta(\partial_\mu\phi_1) = \partial_\mu(\phi_1 + \delta\phi_1) - \partial_\mu(\phi_1) = \partial_\mu(\delta\phi_1)$. The first term can be rewritten to $\partial^\mu[(\partial_\mu\phi_1)\delta\phi_1] - \square\phi_1(\delta\phi_1)$. After substitution into the action integral $\delta S = \int d^4x \delta\mathcal{L}$, the term $\partial^\mu[(\partial_\mu\phi_1)\delta\phi_1]$ leads to a surface integral of $\partial_\mu\phi_1(\delta\phi_1)$, by the divergence theorem in four dimensions. Thus if we restrict ourselves to variations $\delta\phi_i$ that vanish at the boundaries, we can ignore the term.

This gives

$$\delta S = - \int d^4x [\square\phi_1 + m_1^2\phi_1 + 2\lambda\phi_1\phi_2^2]\delta\phi_1.$$

Requiring δS to vanish for an arbitrary variation $\delta\phi_1$ leads to the equation of motion (EOM) for ϕ_1 :

$$\square\phi_1 + m_1^2\phi_1 + 2\lambda\phi_1\phi_2^2 = 0.$$

As the Lagrangian is symmetric under the interchange of field labels $1 \leftrightarrow 2$, the EOM for ϕ_2 immediately follows:

$$\square\phi_2 + m_2^2\phi_2 + 2\lambda\phi_1^2\phi_2 = 0.$$

The Lagrangian term $-\lambda\phi_1^2\phi_2^2$ couples the two fields, leading to ϕ_2 showing up in the EOM for ϕ_1 and vice versa, meaning that they interact. Klein-Gordon equations for both fields are retrieved in the limit where the coupling vanishes ($\lambda \rightarrow 0$).

b) Determine the Feynman rules of the theory. Ignore combinatorics / symmetry factors. You may take the usual Feynman rules for ϕ^4 theory for granted. Also recall the formula

$$\langle \mathbf{p}'\mathbf{k}' | S | \mathbf{p}\mathbf{k} \rangle = {}_0\langle \mathbf{p}'\mathbf{k}' | T e^{-i \int dt H_I(t)} | \mathbf{p}\mathbf{k} \rangle_{0 \text{ connected, amputated}}, \quad (4)$$

where “0” stands for free eigenstates, $\mathbf{p}$, $\mathbf{k}$, $\mathbf{p}'$, and $\mathbf{k}'$ are the momenta of any of the two particle types, and $H_I(t)$ is the interaction Hamiltonian written in the interaction picture.

Give an example of an $\mathcal{O}(\lambda^2)$ process, in terms of a Feynman diagram and the corresponding expression for $i\mathcal{M}$. (Leave any momentum integrals undetermined.)

Solution: The interaction Hamiltonian is (to obtain the Hamiltonian from the Lagrangian see formula collection):

$$H_I(t) = \lambda \int d^3x \phi_1^2 \phi_2^2, \quad (5)$$

where ϕ_1 and ϕ_2 now are interaction picture fields. Expanding the exponential, and consider e.g. the second order:

$$\langle \mathbf{p}' \mathbf{k}' | S | \mathbf{p} \mathbf{k} \rangle = (-i\lambda)^2 \langle \mathbf{p}' \mathbf{k}' | T \int d^4x \phi_1 \phi_1 \phi_2 \phi_2 \int d^4y \phi_1 \phi_1 \phi_2 \phi_2 | \mathbf{p} \mathbf{k} \rangle_0, \quad (6)$$

where still “connected, amputated” is implied. First we note that the propagators of the two fields are the same as for a single scalar field, except that the masses are m_1 and m_2 , respectively. This is because the propagator is derived from the free field.

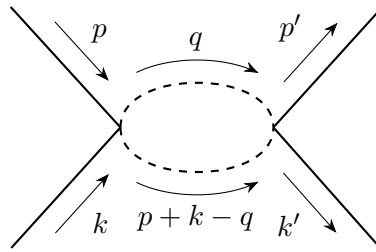
Now we use Wick’s theorem to express (6) in terms of contractions. Contractions between different fields, or between ϕ_1 and an external state $|\mathbf{p}\rangle$ for field 2, or vice versa, will give zero, since ladder operators associated with field 1 commute with ladder operators for field 2.

A contraction involving equal fields equals the corresponding propagator. Contractions between a field and the corresponding external state will be exactly as for ϕ^4 theory.

Since the integrand of the integrals in (6) consists of two ϕ_1 fields and two ϕ_2 fields, each vertex will consist of two field-1 lines and two field-2 lines. Let, for example, the field-1 lines be solid, while field-2 lines are dashed.

This gives us the same Feynman rules as for ϕ^4 theory, except that there are two different lines (and propagators) involved, and with vertices consisting of two of each type of line.

An example of an $\mathcal{O}(\lambda^2)$ -process: $\phi_1 \phi_1 \rightarrow \phi_1 \phi_1$, given by the diagram



The expression is

$$i\mathcal{M} = (-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_2^2 + i\epsilon} \cdot \frac{i}{(p+k-q)^2 - m_2^2 + i\epsilon}. \quad (7)$$

Problem 3 Scalar decay in Yukawa theory

This problem explores the decay of the scalar to a fermion-antifermion pair in Yukawa theory (with Lagrangian and Feynman rules listed in the formula collection). This is actually a very important process in the Standard Model, as the (scalar) Higgs boson interaction with fermions is described by exactly such a theory, and in particular $h \rightarrow b\bar{b}$ (where h is the Higgs boson, and b is the second-heaviest quark) is the dominant decay channel for the Higgs boson. Let M be the mass of the scalar, and m the mass of each of the fermions.

a) Draw the leading-order (first order in the coupling parameter g) Feynman diagram for this decay process, and write down its amplitude.

Solution: As it contains no propagators, the leading-order diagram is simply given by

$$i\mathcal{M} = \text{---} \text{---} \text{---} \begin{array}{c} \nearrow p_1 \\ \searrow p_2 \end{array} = -ig\bar{u}^s(p_1)v^r(p_2), \quad (8)$$

where $p_1 + p_2 = (M, \mathbf{0})$ in the center-of-mass frame, and s, r are the spin states of the fermion and antifermion, respectively.

b) Calculate the leading-order decay rate Γ of the scalar, assuming that we do not measure the spin of the final-state fermions. (See formula collection.)

Solution: The fact that the final-state spins are not measured suggests that they should be summed over. The squared matrix element is then

$$\begin{aligned} |\mathcal{M}|^2 &= g^2 \sum_{s,r=1}^2 \bar{u}^s(p_1)v^r(p_2)\bar{v}^r(p_2)u^s(p_1) \\ &= g^2 \text{Tr}[(\not{p}_1 + m)(\not{p}_2 - m)]. \end{aligned} \quad (9)$$

The traces evaluate to, using the identities in the formula collection, $\text{Tr}[\not{p}_1 \not{p}_2] = 4p_1 \cdot p_2$, $\text{Tr}[-m^2] = -4m^2$ (recall that there is an implicit 4×4 identity matrix multiplying all numbers in expressions such as these). Using $2p_1 \cdot p_2 = (p_1 + p_2)^2 - p_1^2 - p_2^2 = M^2 - 2m^2$ and collecting terms, we find

$$|\mathcal{M}|^2 = 2g^2 M^2 \left(1 - \frac{4m^2}{M^2}\right). \quad (10)$$

The differential decay rate is, using the expression in the formula collection as well as the simplification for two-particle phase space, with $E_{\text{cm}} = M$,

$$d\Gamma = \frac{|\mathbf{p}_1|}{32\pi^2 M^2} |\mathcal{M}|^2 d\Omega, \quad (11)$$

with $|\mathbf{p}_1|^2 = E_1^2 - m^2 = \frac{M^2}{4} - m^2 = \frac{M^2}{4} \left(1 - \frac{4m^2}{M^2}\right)$. The integral over the solid angles Ω is trivial in this case, since there is no angular dependence in the squared matrix element (as an aside, this is due to the spinless nature of the scalar), just giving a factor $\int d\Omega = 4\pi$. The decay rate is then

$$\Gamma = \frac{g^2 M}{8\pi} \left(1 - \frac{4m^2}{M^2}\right)^{3/2}. \quad (12)$$

c) Argue that the analogous QED process, $\gamma \rightarrow e^+e^-$, is not possible.

Solution: For the electron-positron pair, one can find a reference frame where the total 3D-momentum is zero. For the photon, on the other hand, the 3D-momentum is nonzero in any frame. Thus the suggested process does not satisfy momentum conservation.

In more detail, if $q = p_1 + p_2$, where q, p_1, p_2 are the four-momenta of the photon, the electron and the positron, respectively, we must have $0 = q^2 = 2m^2 + 2p_1 p_2$. This is a Lorentz invariant equation. Going to the frame where $\mathbf{p}_1 = -\mathbf{p}_2$, we see that the equation cannot be satisfied.

Problem 4 LSZ reduction

In this problem we consider a self-interacting, scalar theory with field $\phi(x)$. Define operators

$$a_{\mathbf{p}}^{\psi}(t) = i \int d^3x \Psi_{\mathbf{p}}^*(x) \overleftrightarrow{\partial}_0 \phi(x), \quad (13a)$$

$$a_{\mathbf{p}}^{\psi\dagger}(t) = -i \int d^3x \Psi_{\mathbf{p}}(x) \overleftrightarrow{\partial}_0 \phi(x), \quad (13b)$$

where

$$\Psi_{\mathbf{p}}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{\psi(\mathbf{k} - \mathbf{p})}{\sqrt{2E_{\mathbf{k}}}} e^{-ikx}, \quad k^0 = E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}, \quad (14)$$

with the single particle mass m in the interacting theory. Here $\psi(\mathbf{k})$ is a spectrum centered about $\mathbf{k} = 0$, e.g., a gaussian.

a) Taking ϕ^4 theory as an example, use

$$\begin{aligned} & \langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle \\ &= \frac{\langle 0 | T \{ \phi_I(x_1) \cdots \phi_I(x_n) e^{-i \int dt H_I(t)} \} | 0 \rangle}{\langle 0 | T \{ e^{-i \int dt H_I(t)} \} | 0 \rangle}, \quad n \geq 1, \end{aligned} \quad (15)$$

to argue that

$$\langle \Omega | \phi(x) | \Omega \rangle = 0. \quad (16)$$

Here T denotes time ordering, $H_I(t)$ is the interaction Hamiltonian written in the interaction picture, and $\phi_I(x)$ is the field in the interaction picture.

Solution: From the expression for the Hamiltonian in terms of the Lagrangian, and the Lagrangian of ϕ^4 theory (formula sheet), we have

$$H_I(t) = \frac{\lambda}{4!} \int d^3x \phi_I^4(x) \quad (17)$$

Using the formula (15) for $n = 1$, and noting that the expanded exponential has only an even number of terms, we find that the numerator $\langle 0 | T \{ \phi_I(x_1) \cdots \phi_I(x_n) e^{-i \int dt H_I(t)} \} | 0 \rangle$ contains an odd number of fields for all terms (1 field, 5 fields, 9 fields, etc.). Thus after using Wick's theorem, all terms will contain normal ordered products of uncontracted field(s). It follows that the vacuum expectation value is zero.

b) Let $|\lambda_{\mathbf{p}}\rangle$ be the eigenstates of the total Hamiltonian H and the momentum operator $\mathbf{P}$, with eigenvalues $p^0 = E_{\mathbf{p}}^{\lambda} \equiv \sqrt{\mathbf{p}^2 + m_{\lambda}^2}$ and $\mathbf{p}$, respectively. Here $m_{\lambda} > 0$ is the energy eigenvalue in the rest frame. From Poincaré symmetry we have

$$\langle \Omega | \phi(x) | \lambda_{\mathbf{p}} \rangle = \langle \Omega | \phi(0) | \lambda_0 \rangle e^{-ipx}, \quad p^0 = E_{\mathbf{p}}^{\lambda}. \quad (18)$$

Prove that

$$\langle \Omega | a_{\mathbf{p}}^{\psi}(t) | \Omega \rangle = 0. \quad (19)$$

Also calculate the matrix element

$$\langle \lambda_{\mathbf{p}} | a_{\mathbf{q}}^{\psi}(\pm\infty) | \Omega \rangle \quad (20)$$

and argue that it vanishes if $\langle \lambda_{\mathbf{p}} |$ is replaced by a (possibly narrow-banded) wavepacket (superposition of $\langle \lambda_{\mathbf{p}} |$ for a narrow band of $\mathbf{p}$'s).

From this argue that

$$a_{\mathbf{q}}^{\psi}(\pm\infty) | \Omega \rangle = 0. \quad (21)$$

Solution: Eq. (19) follows directly from the definition of the operator, and (16). Furthermore,

$$\begin{aligned} \langle \lambda_{\mathbf{p}} | a_{\mathbf{q}}^{\psi}(t) | \Omega \rangle &= i \int d^3x \Psi_{\mathbf{q}}^*(x) \overset{\leftrightarrow}{\partial}_0 \langle \lambda_{\mathbf{p}} | \phi(x) | \Omega \rangle \\ &= \langle \lambda_0 | \phi(0) | \Omega \rangle \int d^3x \Psi_{\mathbf{q}}^*(x) i \overset{\leftrightarrow}{\partial}_0 e^{ipx} \Big|_{p^0=E_{\mathbf{p}}^{\lambda}} \\ &= \langle \lambda_0 | \phi(0) | \Omega \rangle \int \frac{d^3k}{(2\pi)^3} \frac{\psi^*(\mathbf{k}-\mathbf{q})}{\sqrt{2E_{\mathbf{k}}}} \int d^3x e^{ikx} i \overset{\leftrightarrow}{\partial}_0 e^{ipx} \Big|_{p^0=E_{\mathbf{p}}^{\lambda} \text{ and } k^0=E_{\mathbf{k}}} \\ &= \langle \lambda_0 | \phi(0) | \Omega \rangle \int \frac{d^3k}{(2\pi)^3} \psi^*(\mathbf{k}-\mathbf{q}) \frac{E_{\mathbf{k}} - E_{\mathbf{p}}^{\lambda}}{\sqrt{2E_{\mathbf{k}}}} e^{i(E_{\mathbf{k}}+E_{\mathbf{p}}^{\lambda})t} (2\pi)^3 \delta(\mathbf{p}+\mathbf{k}) \\ &= \langle \lambda_0 | \phi(0) | \Omega \rangle \cdot \psi^*(-\mathbf{p}-\mathbf{q}) \frac{E_{\mathbf{p}} - E_{\mathbf{p}}^{\lambda}}{\sqrt{2E_{\mathbf{p}}}} e^{i(E_{\mathbf{p}}+E_{\mathbf{p}}^{\lambda})t}. \end{aligned} \quad (22)$$

When λ corresponds to a single particle, this expression clearly vanishes. But if not, it also vanishes provided we form a (possibly narrow-banded)

superposition of $\mathbf{p}$'s. This is because for $t = \pm\infty$ the exponential oscillates rapidly under the $\mathbf{p}$ -integral such that the integral is washed out.

Eq. (21) now follows, since from (19) and the vanishing of (20), $a_{\mathbf{q}}^{\psi}(\pm\infty)|\Omega\rangle$ is orthogonal to all vectors of a basis; thus it is the zero vector.

c) Derive an expression for $\langle\Omega|\phi(y)a_{\mathbf{k}}^{\psi\dagger}(-\infty)|\Omega\rangle$ in terms of the two-point correlation function $\langle\Omega|T\phi(y)\phi(x)|\Omega\rangle$. The result should be as simple as possible. It should also be written in four-dimensional (manifestly covariant) form, except for the presence of $\Psi_{\mathbf{k}}(x)$. *Hint:* Use (21) to insert a term with another creation operator.

Solution: We use (21) to insert another term with the creation operator to the left of the field:

$$\begin{aligned}\langle\Omega|\phi(y)a_{\mathbf{k}}^{\psi\dagger}(-\infty)|\Omega\rangle &= \langle\Omega|\phi(y)a_{\mathbf{k}}^{\psi\dagger}(-\infty)|\Omega\rangle - \langle\Omega|a_{\mathbf{k}}^{\psi\dagger}(\infty)\phi(y)|\Omega\rangle \\ &= i \left[\lim_{t \rightarrow \infty} - \lim_{t \rightarrow -\infty} \right] \int d^3x \Psi_{\mathbf{k}}(x) \overset{\leftrightarrow x}{\partial}_0 \langle\Omega|T\phi(y)\phi(x)|\Omega\rangle \\ &= i \int d^4x \Psi_{\mathbf{k}}(x) (\square_x + m^2) \langle\Omega|T\phi(y)\phi(x)|\Omega\rangle.\end{aligned}\quad (23)$$

Here we have used that if $f(x)$ satisfies the Klein-Gordon equation,

$$\begin{aligned}\left[\lim_{t \rightarrow \infty} - \lim_{t \rightarrow -\infty} \right] \int d^3x f(x) \overset{\leftrightarrow}{\partial}_0 g(x) &= \int d^4x \partial_0 \left(f(x) \overset{\leftrightarrow}{\partial}_0 g(x) \right) \\ &= \int d^4x [f(x) \partial_0^2 g(x) - (\partial_0^2 f(x)) g(x)] \\ &= \int d^4x [f(x) \partial_0^2 g(x) + [(m^2 - \nabla^2) f(x)] g(x)] \\ &= \int d^4x f(x) (\square + m^2) g(x).\end{aligned}\quad (24)$$

In the last step we have use integration by parts twice, to move the ∇^2 from $f(x)$ to $g(x)$. The associated surface terms vanish when $f(x)$ and therefore $\Psi_{\mathbf{k}}(x)$ tend sufficiently fast to zero in 3D space for fixed x^0 .

d) Why is the LSZ reduction in the previous question (generalized to more than two fields) useful?

Solution: The ladder operator(s) on the left-hand side of the equation can be used to express the in- and out-states in scattering experiments. Thus we can calculate the S matrix from correlation functions, which in turn can be expressed as a sum of Feynman diagrams.

Formulas

Lagrangian field theory:

$$\begin{aligned}
 S &= \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \\
 \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= 0 \\
 \pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
 H &= \int d^3x (\pi \dot{\phi} - \mathcal{L})?
 \end{aligned}$$

Noether:

$$\begin{aligned}
 \phi_r &\mapsto \phi_r + \alpha \Delta \phi_r \\
 \mathcal{L} &\mapsto \mathcal{L} + \alpha \partial_\mu J^\mu \\
 j^\mu &= \sum_r \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \Delta \phi_r - J^\mu
 \end{aligned}$$

Lorentz transformations, boost along the z -axis:

$$\begin{aligned}
 t' &= \gamma(t + \beta z) \\
 z' &= \gamma(z + \beta t)
 \end{aligned}$$

Klein-Gordon field:

$$\begin{aligned}
 \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right) \\
 [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}).
 \end{aligned}$$

Dirac field:

$$\begin{aligned}
 \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_r \left(a_{\mathbf{p}}^r u^r(\mathbf{p}) e^{-ipx} + b_{\mathbf{p}}^{r\dagger} v^r(\mathbf{p}) e^{ipx} \right) \\
 u^s(p) &= \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix}, \quad v^s(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix} \\
 \bar{u}^r(p) u^s(p) &= -\bar{v}^r(p) v^s(p) = 2m \delta^{rs} \\
 \sum_{s=1}^2 u^s(p) \bar{u}^s(p) &= \not{p} + m, \quad \sum_{s=1}^2 v^s(p) \bar{v}^s(p) = \not{p} - m
 \end{aligned}$$

Chiral representation:

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}.$$

Definitions:

$$f \overset{\leftrightarrow}{\partial}_0 g = f \partial_0 g - (\partial_0 f) g$$

Identities:

$$\begin{aligned}
\sigma^i \sigma^j &= \delta^{ij} + i\epsilon^{ijk} \sigma^k \\
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\
\gamma^\mu \gamma_\mu &= 4 \\
\gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\
\gamma^\mu \gamma^\nu \gamma^\alpha \gamma_\mu &= 4g^{\nu\alpha} \\
\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma_\mu &= -2\gamma^\rho \gamma^\alpha \gamma^\nu \\
\text{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu} \\
\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\rho} g^{\nu\sigma} + 4g^{\mu\sigma} g^{\nu\rho} \\
\text{Tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] &= 0, \text{ for } n \text{ odd}
\end{aligned}$$

Cross-sections and decay rates:

$$\begin{aligned}
d\sigma &= \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\
&\quad \times |\mathcal{M}(p_A, p_B \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A + p_B - \sum p_f) \\
d\Gamma &= \frac{1}{2m_A} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\
&\quad \times |\mathcal{M}(p_A = (m_A, \mathbf{0}) \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A - \sum p_f)
\end{aligned}$$

Two-particle phase space:

$$\int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2) = \int d\Omega_{\text{cm}} \frac{1}{16\pi^2} \frac{|\mathbf{p}_1|}{E_{\text{cm}}},$$

$E_{\text{cm}} = E_1 + E_2$ in the center-of-mass frame; Ω_{cm} = solid angle for particle 1. For decay of a single particle, set $p_B = 0$.

Lagrangians and Feynman rules

The following rules are used to obtain the amplitude $i\mathcal{M}$, and given in momentum space. Dots represent vertices in all of the diagrams listed below. For consistency between ϕ^4 theory and Yukawa theory, scalar particles are represented by dashed lines, and their masses by M , in both cases.

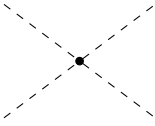
Common for all theories:

- Impose momentum conservation at each vertex
- Integrate over each undetermined momentum, $\int \frac{d^4 k}{(2\pi)^4}$

ϕ^4 theory

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

- Propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

- External line: $\xrightarrow{p} \bullet = 1$
- Vertex:  $= -i\lambda$
- Divide by the symmetry factor

Yukawa theory:

$$\mathcal{L}_Y = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2 \phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\psi\phi$$

- Scalar propagator: $\xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$;
- Fermion propagator: $\xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$;
- External scalar: $\xrightarrow{p} \bullet = 1$
- Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$
- Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$
- Outgoing fermion (spin s): $\xrightarrow{p} \bullet = \bar{u}^s(p)$
- Outgoing antifermion (spin s): $\xleftarrow{p} \bullet = v^s(p)$
- Vertex:  $= -ig$
- Work out the overall sign of the diagram

QED:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad e = -|e|$$

- Photon propagator: $\mu \xrightarrow{p} \nu = \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}$
- Fermion propagator: $\xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$;
- Incoming photon: $\mu \xrightarrow{p} \bullet = \epsilon_\mu(p)$

- Outgoing photon: $\bullet \xrightarrow{p} \mu = \epsilon_\mu^*(p)$
- Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$
- Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$
- Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$
- Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$
- Vertex: $\mu \text{ ~~~~~ } \bullet \begin{array}{l} \nearrow \\ \searrow \end{array} = -ie\gamma^\mu$
- Work out the overall sign of the diagram