

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: FYS4170 — Relativistic Quantum Field Theory

Day of examination: 27. november 2023

Examination hours: 15.00 – 19.00

This problem set consists of 12 pages.

Appendices: Formula sheets are included.

Permitted aids: No examination support material is allowed.

Make sure that your copy of this examination paper is complete before answering.

Problem 1

In the course we have used ϕ^4 theory as a relatively simple entry point to many of the methods introduced for interacting quantum field theories. A theory that appears to be even simpler, which also leads to self-interactions, is ϕ^3 theory, described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}M^2\phi^2 - \frac{g}{3!}\phi^3. \quad (1)$$

a) Find the mass dimension n of the coupling g ($g \sim (\text{mass})^n$ in natural units).

Solution:

- Space and time must have dimension -1 . This can for example be seen from (1); $\partial_\mu = \partial/\partial x^\mu$ must have the same dimension as M , i.e. 1.
- The action $S = \int d^4x \mathcal{L}$ is dimensionless since it has dimension energy $\cdot$ time; thus $\mathcal{L}$ has mass dimension 4.
- ϕ then has mass dimension 1 (from the first two terms of (1)).
- Finally, this means that g must have mass dimension 1.

b) The reason why we usually do not work with this theory can be seen from its Hamiltonian H . Find the expression for H , and explain why (1) cannot describe a physical theory (recall that H describes the energy of a system).

Solution: To compute $H = \int d^3x (\pi \dot{\phi} - \mathcal{L})$ we first need the conjugate momentum π , which is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \partial^0 \phi = \dot{\phi}.$$

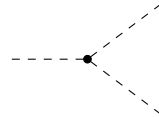
With this the Hamiltonian becomes

$$\begin{aligned} H &= \int d^3x \left[\pi^2 - \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} M^2 \phi^2 + \frac{g}{3!} \phi^3 \right] \\ &= \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} M^2 \phi^2 + \frac{g}{3!} \phi^3 \right]. \end{aligned}$$

The issue with this expression, if this were to describe an actual, physical theory, is that the potential energy (described by the last two terms) is unbounded from below. This is seen from the fact that the last term is arbitrarily negative and may dominate. This in turn makes the kinetic energy arbitrarily large, meaning that the system described by the theory is severely unstable.

Ignoring this problem for a moment, we can take a look at the processes that are possible in this theory.

c) Determine the Feynman rules compared to those of ϕ^4 theory, and argue in particular why the interaction vertex is



$$= -ig. \quad (2)$$

You may ignore symmetry factors and combinatorics. Recall the formula

$$\langle \mathbf{p}' \mathbf{k}' | S | \mathbf{p} \mathbf{k} \rangle = {}_0 \langle \mathbf{p}' \mathbf{k}' | T e^{-i \int dt H_I(t)} | \mathbf{p} \mathbf{k} \rangle_{0 \text{ connected, amputated}}, \quad (3)$$

where “0” stands for free eigenstates, $\mathbf{p}$, $\mathbf{k}$, $\mathbf{p}'$, and $\mathbf{k}'$ are the momenta of external states, and $H_I(t)$ is the interaction part of the Hamiltonian written in the interaction picture.

Solution: We have

$$H_I(t) = \frac{g}{3!} \int d^3x \phi_I^3, \quad (4)$$

where ϕ_I is the field in the interaction picture. Expanding the exponential, and consider e.g. the second order:

$$\langle \mathbf{p}' \mathbf{k}' | S | \mathbf{p} \mathbf{k} \rangle = (-ig)^2 {}_0 \langle \mathbf{p}' \mathbf{k}' | T \int d^4x \phi_I \phi_I \phi_I \int d^4y \phi_I \phi_I \phi_I | \mathbf{p} \mathbf{k} \rangle_0, \quad (5)$$

where still “connected, amputated” is implied. With the help of Wick’s theorem we get contractions exactly as for ϕ^4 theory; the propagator is the same since the free part of the Hamiltonian is the same. Also external line contractions are the same. Since the integrand of the integrals in (5) consists of three fields, each vertex will consist of three lines.

d) The simplest process we can imagine is $\phi(p) \rightarrow \phi(p_1)\phi(p_2)$ (in words, this means one scalar particle in the initial state and two in the final state, with momenta in parentheses). The leading-order Feynman diagram is given by the vertex in (2). This process is usually kinematically forbidden, except for a specific combination of Lagrangian parameters and configuration of momenta p_1, p_2 . Find this combination.

Solution: Firstly, the process is impossible for $M \neq 0$ by energy conservation: In the rest frame of the initial-state particle the initial-state energy is $E = M$, while the final-state energy is $E_1 + E_2 \geq 2M$. Thus we must have $M = 0$ for this to be a viable process.

For $M = 0$, conservation of 4-momentum gives $p = p_1 + p_2$, with $p^2 = p_1^2 = p_2^2 = 0$. Squaring both sides of this equation then gives

$$0 = 2p_1 \cdot p_2 = 2(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2).$$

Using $E_i = |\mathbf{p}_i|$ for massless particles, as well as $\mathbf{p}_1 \cdot \mathbf{p}_2 = |\mathbf{p}_1||\mathbf{p}_2| \cos \alpha$, where α is the angle between the two final-state particles, this leaves

$$0 = 2|\mathbf{p}_1||\mathbf{p}_2|(1 - \cos \alpha),$$

meaning that we must have $\alpha = 0$.

Thus the only way for this process to be possible is with $M = 0$ and $\mathbf{p}_1$ and $\mathbf{p}_2$ in the same direction. In other words the two final-state particles move exactly in parallel, and with the same combined momentum as the original particle.

e) A more interesting process is the scattering process $\phi(p_1)\phi(p_2) \rightarrow \phi(p_3)\phi(p_4)$, which is always possible. Draw all diagrams that contribute to this process to lowest order.

For the case $M = 0$, introduce Mandelstam variables,

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \quad (6)$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2, \quad (7)$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2. \quad (8)$$

where $s + t + u = 0$, and show that the spin-averaged, differential cross-section¹

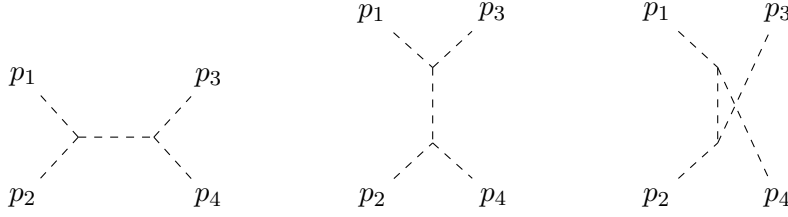
$$\frac{d\sigma}{dt} = \frac{1}{32\pi s^2} \langle |\mathcal{M}|^2 \rangle, \quad (9)$$

is given by

$$\frac{d\sigma}{dt} = \frac{g^4}{32\pi} \frac{[s^2 + t^2 + st]^2}{s^4 t^2 (s + t)^2}. \quad (10)$$

¹There is a factor 1/2 in this expression relative to a similar one that was derived in the exercises, which appears since the final state contains two indistinguishable particles.

Solution: There are three contributing diagrams:



The form of their corresponding matrix elements is reasonably simple:

$$\begin{aligned}
 i\mathcal{M} &= -ig^2 \left[\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right] \\
 &= -ig^2 \frac{tu + su + st}{stu} \\
 &= -ig^2 \frac{s^2 + t^2 + st}{st(s+t)},
 \end{aligned}$$

where we have used $u = -s - t$.

The spin-averaged squared amplitude follows immediately by simply squaring the above equation, since scalars have only one possible spin state; inserting this in (9) leads immediately to

$$\frac{d\sigma}{dt} = \frac{g^4}{32\pi} \frac{[s^2 + t^2 + st]^2}{s^4 t^2 (s+t)^2}.$$

Problem 2

a) The Lagrangian for the Dirac field ψ is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \quad (11)$$

Find the Euler-Lagrange equation for ψ and $\bar{\psi}$ by treating ψ and $\bar{\psi}$ as independent.

Solution: The Euler-Lagrange equation for a field ϕ is

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

We get one equation for each field ψ_β :

$$\partial_\mu \bar{\psi}_\alpha i\gamma^\mu_{\alpha\beta} + m\bar{\psi}_\beta = 0, \quad (12)$$

and one for each field $\bar{\psi}_\alpha$:

$$i\gamma^\mu_{\alpha\beta} \partial_\mu \psi_\beta - m\psi_\alpha = 0. \quad (13)$$

The latter equation can be written

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0, \quad (14)$$

which is the Dirac equation, while the former can be shown by properties of the γ matrices ($(\gamma^0)^2 = 1$, $\gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger$) to be Dirac equation in Hermitian-conjugate form.

b) The S -matrix elements are usually written in two equivalent ways:

$${}_{\text{out}}\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | \mathbf{k}_1 \mathbf{k}_2 \cdots \rangle_{\text{in}} = \langle \mathbf{p}_1 \mathbf{p}_2 \cdots | S | \mathbf{k}_1 \mathbf{k}_2 \cdots \rangle, \quad (15)$$

The reference time for the Schrödinger and Heisenberg picture is set at remote past $t = -T$. Explain what the 6 quantities $|\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle$, $|\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle_{\text{in}}$, S , $S|\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle$, $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle$, $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle_{\text{out}}$ are, relate state vectors in the two pictures, and from the relations prove the equality (15).

Solution: $|\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle$ and $|\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle_{\text{in}}$ are the initial states in the Schrödinger and Heisenberg picture, respectively, but since the pictures coincide at the reference time $t = -T$ the two state vectors are equal:

$$|\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle = |\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle_{\text{in}}. \quad (16)$$

The operator S is the unitary time-evolution operator in the Schrödinger picture (or Heisenberg picture), so the state $S|\mathbf{k}_1 \mathbf{k}_2 \cdots\rangle$ is the in-state evolved to time $t = T$.

The state vectors $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle$ and $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle_{\text{out}}$ are another pair describing a so-called out-state; $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle$ is in the Schrödinger picture at $t = T$, and $|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle_{\text{out}}$ is in the Heisenberg picture:

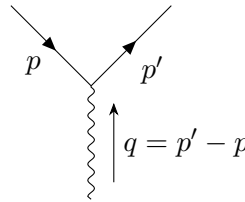
$$|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle = S|\mathbf{p}_1 \mathbf{p}_2 \cdots\rangle_{\text{out}}. \quad (17)$$

We need this out-state to describe transitions (from the in-state to the out-state).

We can now establish (15):

$$\begin{aligned} {}_{\text{out}}\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | \mathbf{k}_1 \mathbf{k}_2 \cdots \rangle_{\text{in}} &= {}_{\text{out}}\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | S^\dagger S | \mathbf{k}_1 \mathbf{k}_2 \cdots \rangle_{\text{in}} \\ &= \langle \mathbf{p}_1 \mathbf{p}_2 \cdots | S | \mathbf{k}_1 \mathbf{k}_2 \cdots \rangle. \end{aligned} \quad (18)$$

c) Consider a vertex in QED with an external state electron as input and as output, given at tree-level by



$$= -ie\gamma^\mu. \quad (19)$$

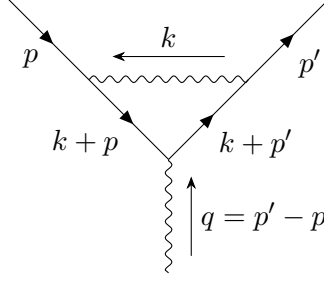
Draw the lowest order vertex correction in quantum electrodynamics (QED).

The vertex correction (to any order) amounts to substitute $\gamma^\mu \mapsto \Gamma^\mu$ in the vertex rule. Argue that Γ^μ (to any order) can be written in the form

$$\Gamma^\mu = \gamma^\mu A + (p^\mu + p'^\mu)B, \quad (20)$$

where the coefficients A and B only depend on q^2 (in addition to constants such as m and e). You may assume that A is a number.

Solution: Lowest order vertex correction:



where the loop momentum k is to be integrated over.

When computing the full diagram, including external states, Γ^μ is to be sandwiched between $\bar{u}(p')$ and $u(p)$; then it becomes a four-vector. Moreover, from the Feynman rules it is expressed from p , p' , γ^μ , e , m , and pure numbers. Thus we can write

$$\Gamma^\mu = \gamma^\mu A + (p^\mu + p'^\mu)B + (p'^\mu - p^\mu)C, \quad (21)$$

where the coefficients A , B , C may depend on constants (m , e) and $q^2 = (p' - p)^2 = 2m^2 - 2pp'$. The other Lorentz scalars $p^2 = m^2$, $p'^2 = m^2$, $pq = pp' - m^2$, $p'q = m^2 - pp'$ do not add anything. The coefficients could contain $\not{p}$, $\not{p}'$, but since $\not{p}u(p) = mu(p)$ and $\bar{u}(p')\not{p}' = \bar{u}(p')m$ (from the Dirac equation), this does not add anything either.

Γ^μ describes a sum of diagrams with the photon line removed. Thus we can apply Ward's identity $q_\mu \bar{u}(p')\Gamma^\mu u(p) = 0$:

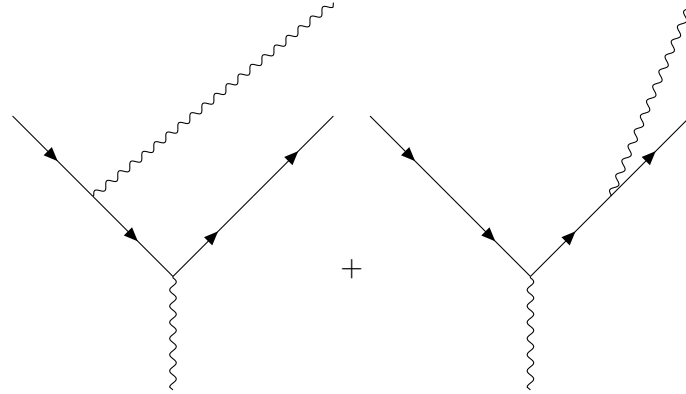
$$\begin{aligned} 0 &= \bar{u}(p')q_\mu \Gamma^\mu u(p) \\ &= \bar{u}(p')\not{q}u(p)A + \bar{u}(p')(p'^2 - p^2)Bu(p) + \bar{u}(p')q^2Cu(p) \\ &= \bar{u}(p')Cu(p)q^2, \end{aligned} \quad (22)$$

where we have used the Dirac equation and $p^2 = m^2 = p'^2$. This is to be valid for any electron momenta p , p' . In general $\bar{u}(p')u(p) \neq 0$; this is in particular easy to see if p , p' are nonrelativistic. Thus $C = 0$, and we obtain (20).

d) When calculating the cross section associated with lowest order vertex correction, it turns out that the result is infrared divergent. What is the problem, and how is it solved?

Solution: Infrared divergence occurs because we ask the wrong question: The process is not possible to distinguish from a process with

low-energy bremsstrahlung, shown below,



and after adding the cross sections the divergence disappears. Put differently, one must include all terms to the same order in α .

Problem 3

a) Let $\mathcal{V}$ be the set of all different, disconnected Feynman diagram pieces. Let A be the value of a particular diagram, consisting of a connected piece and (possibly) several disconnected pieces. Argue that

$$A = (\text{value of connected piece}) \cdot \prod_i \frac{1}{n_i!} V_i^{n_i}, \quad (23)$$

where V_i for $i = 1, 2, \dots$ are the (values of the) diagrams in $\mathcal{V}$. What is n_i ?

Solution: A is the product of all pieces in the diagram. For example, A may consist of the connected piece times 2 copies of the first element in $\mathcal{V}$, and 1 copy of the second element in $\mathcal{V}$; this gives $A = (\text{value of connected piece}) \cdot \frac{V_1^2}{2!} V_2$. Here the denominator $2!$ is included since interchange of the two copies does not change the overall diagram. In general we similarly obtain (23), where n_i denotes the number of copies of the V_i diagram in A , and i indexes the elements in $\mathcal{V}$.

b) Prove that the sum of all diagrams with m external points is

$$\sum (\text{connected diagrams with } m \text{ external points}) \cdot \exp\left(\sum_i V_i\right), \quad (24)$$

where i indexes the elements in $\mathcal{V}$.

Solution: We are interested in the sum of all diagrams with m external points. This is the sum of all possible A 's for a given number of external points, as given in the previous question. We can first find the sum of all diagrams containing the same connected piece as A , by summing (23) over n_i for each i . With

$$\sum_{n_1} \sum_{n_2} \cdots \prod_i \frac{V_i^{n_i}}{n_i!} = \sum_{n_1} \frac{V_1^{n_1}}{n_1!} \sum_{n_2} \frac{V_2^{n_2}}{n_2!} \cdots = \exp(V_1) \exp(V_2) \cdots, \quad (25)$$

we find that the sum of all diagrams containing a particular connected piece is

$$(\text{value of connected piece}) \cdot \exp\left(\sum_i V_i\right). \quad (26)$$

Repeating the same process for every other connected diagram with the same number of external points m , and adding all of these together, we end up with (24).

Formulas

Lagrangian field theory:

$$\begin{aligned}
 S &= \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \\
 \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} &= 0 \\
 \pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
 H &= \int d^3x (\pi \dot{\phi} - \mathcal{L})
 \end{aligned}$$

Noether:

$$\begin{aligned}
 \phi_r &\mapsto \phi_r + \alpha \Delta \phi_r \\
 \mathcal{L} &\mapsto \mathcal{L} + \alpha \partial_\mu J^\mu \\
 j^\mu &= \sum_r \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} \Delta \phi_r - J^\mu
 \end{aligned}$$

Lorentz transformations, boost along the z -axis:

$$\begin{aligned}
 t' &= \gamma(t + \beta z) \\
 z' &= \gamma(z + \beta t)
 \end{aligned}$$

Klein-Gordon field:

$$\begin{aligned}
 \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right) \\
 (\partial^2 + m^2)\phi &= 0 \\
 [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}).
 \end{aligned}$$

Dirac field:

$$\begin{aligned}
 \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_r \left(a_{\mathbf{p}}^r u^r(\mathbf{p}) e^{-ipx} + b_{\mathbf{p}}^{r\dagger} v^r(\mathbf{p}) e^{ipx} \right) \\
 (i\not{\partial} - m)\psi &= 0 \\
 \{a_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}\} &= \{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs} \\
 u^s(p) &= \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix}, \quad v^s(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix} \\
 \bar{u}^r(p) u^s(p) &= -\bar{v}^r(p) v^s(p) = 2m \delta^{rs} \\
 \not{p} u(p) &= m u(p), \quad \bar{u}(p) \not{p} = \bar{u}(p) m, \\
 \not{p} v(p) &= -m v(p), \quad \bar{v}(p) \not{p} = -\bar{v}(p) m \\
 \sum_{s=1}^2 u^s(p) \bar{u}^s(p) &= \not{p} + m, \quad \sum_{s=1}^2 v^s(p) \bar{v}^s(p) = \not{p} - m
 \end{aligned}$$

Chiral representation:

$$\begin{aligned}
 \gamma^0 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}; \\
 \sigma^1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
 \end{aligned}$$

Definitions:

$$\begin{aligned} f \overleftrightarrow{\partial}_0 g &= f \partial_0 g - (\partial_0 f) g \\ \sigma^\mu &\equiv (1, \sigma^i), \quad \bar{\sigma}^\mu \equiv (1, -\sigma^i) \\ \not{a} &\equiv \gamma^\mu a_\mu \text{ (4-vector } a^\mu) \end{aligned}$$

Identities:

$$\begin{aligned} \sigma^i \sigma^j &= \delta^{ij} + i \epsilon^{ijk} \sigma^k \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\ \gamma^5 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \\ \gamma^\mu \gamma_\mu &= 4 \\ \gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma_\mu &= 4g^{\nu\alpha} \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma_\mu &= -2\gamma^\rho \gamma^\alpha \gamma^\nu \\ \text{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu} \\ \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\rho} g^{\nu\sigma} + 4g^{\mu\sigma} g^{\nu\rho} \\ \text{Tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] &= 0, \text{ for } n \text{ odd} \end{aligned}$$

Ward's identity:

Let $i\mathcal{M}_\mu(q)$ be a diagram where a photon line at vertex μ is removed. Then $\mathcal{M}_\mu(q)q^\mu = 0$.

Gordon's identity:

$$\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p), \text{ where } \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \text{ and } q^\mu = p'^\mu - p^\mu.$$

Cross-sections and decay rates:

$$\begin{aligned} d\sigma &= \frac{1}{2E_{\mathcal{A}} 2E_{\mathcal{B}} |v_{\mathcal{A}} - v_{\mathcal{B}}|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum p_f) \\ d\Gamma &= \frac{1}{2m_{\mathcal{A}}} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_{\mathcal{A}} = (m_{\mathcal{A}}, \mathbf{0}) \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_{\mathcal{A}} - \sum p_f) \end{aligned}$$

Two-particle phase space:

$$\int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^4(p_{\mathcal{A}} + p_{\mathcal{B}} - p_1 - p_2) = \int d\Omega_{\text{cm}} \frac{1}{16\pi^2} \frac{|\mathbf{p}_1|}{E_{\text{cm}}},$$

$E_{\text{cm}} = E_1 + E_2$ in the center-of-mass frame; Ω_{cm} = solid angle for particle 1. For decay of a single particle, set $p_{\mathcal{B}} = 0$.

Lagrangians and Feynman rules

The following rules are used to obtain the amplitude $i\mathcal{M}$, and given in momentum space. Dots represent vertices in all of the diagrams listed below. For consistency between ϕ^4 theory and Yukawa theory, scalar particles are represented by dashed lines, and their masses by M , in both cases.

Common for all theories:

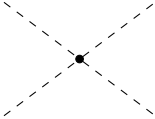
- Impose momentum conservation at each vertex
- Integrate over each undetermined momentum, $\int \frac{d^4k}{(2\pi)^4}$

 ϕ^4 theory

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

• Propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

• External line: $\xrightarrow{p} \bullet = 1$

• Vertex:  $= -i\lambda$

- Divide by the symmetry factor

Yukawa theory:

$$\mathcal{L}_Y = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\psi\phi$$

• Scalar propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

• Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

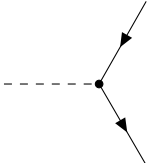
• External scalar: $\xrightarrow{p} \bullet = 1$

• Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$

• Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$

• Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$

• Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$

• Vertex:  $= -ig$

- Work out the overall sign of the diagram

QED:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad e = -|e|$$

- Photon propagator: $\mu \xrightarrow{p} \nu = \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}$
- Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$
- Incoming photon: $\mu \xrightarrow{p} = \epsilon_\mu(p)$
- Outgoing photon: $\bullet \xrightarrow{p} \mu = \epsilon_\mu^*(p)$
- Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$
- Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$
- Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$
- Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$
- Vertex: $\mu \xrightarrow{\quad} \bullet = -ie\gamma^\mu$
- Work out the overall sign of the diagram