

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: FYS4170 — Relativistic Quantum Field Theory

Day of examination: 19. december 2025

Examination hours: 09.00 – 13.00

This problem set consists of 11 pages.

Appendices: Formula sheets are included.

Permitted aids: No examination support material is allowed.

Make sure that your copy of this examination paper is complete before answering.

Problem 1

a) What is a four-vector?

Solution: An object $A^\mu = (A^0, A^1, A^2, A^3)$ consisting of four numbers A^0, A^1, A^2, A^3 , that transforms like four position $x^\mu = (t, x, y, z)$ under Lorentz transformations, i.e., $A^\mu \mapsto \Lambda^\mu_\nu A^\nu$, where Λ is the usual Lorentz transformation matrix.

b) Wick's theorem for two fermion fields is

$$T\psi(x)\bar{\psi}(y) = N\psi(x)\bar{\psi}(y) + \overline{\psi(x)\bar{\psi}(y)}. \quad (1)$$

If (1) is taken as the definition of the contraction $\overline{\psi(x)\bar{\psi}(y)}$, prove the two equalities

$$\begin{aligned} \overline{\psi(x)\bar{\psi}(y)} &= \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle \\ &= \begin{cases} \{\psi^+(x), \bar{\psi}^-(y)\}, & x^0 > y^0, \\ -\{\bar{\psi}^+(y), \psi^-(x)\}, & x^0 < y^0, \end{cases} \end{aligned} \quad (2)$$

where $\psi^+(x)$ and $\psi^-(x)$ are the positive and negative frequency parts of $\psi(x)$.

Solution: The first equality follows directly by taking the vacuum expectation value and noting that the vacuum expectation value of normal ordered expressions vanish. The second equality follows by writing out the positive and negative parts of the fields, and comparing the two sides. Let first $x^0 > y^0$:

$$\begin{aligned} \langle 0|(\psi^+(x) + \psi^-(x))(\bar{\psi}^+(y) + \bar{\psi}^-(y))|0\rangle &= \langle 0|\psi^+(x)\bar{\psi}^-(y)|0\rangle \\ &= \langle 0|\psi^+(x)\bar{\psi}^-(y) + \bar{\psi}^-(y)\psi^+(x)|0\rangle = \{\psi^+(x), \bar{\psi}^-(y)\}. \end{aligned}$$

Here we have used that the positive-frequency parts contain only annihilation operators, and the negative-frequency parts contain only creation operators. Spinor indices are understood in the expressions. For $x^0 < y^0$ we swap the order and insert a minus (remember the definition of time ordering for fermionic fields). After that, the calculation is the same.

c) What is renormalization in quantum field theory? What does it mean that a theory is renormalizable? (About 5-10 sentences.)

Problem 2

In this problem we will investigate a version of QED that contains both a charged fermion and scalar boson, both of which are coupled to the photon field. Assuming all particles to be *massless*, the Lagrangian of the theory is given by

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) + \bar{\psi}(i\not{D})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (3)$$

where $D_\mu = \partial_\mu + ieA_\mu$.

a) Derive the Euler-Lagrange equations for $\phi^\dagger$ and $\bar{\psi}$ (which become equations of motion for ϕ and ψ). Write the result with the covariant derivative D_μ .

Solution: Writing out the covariant derivative D_μ explicitly, the Lagrangian reads

$$\begin{aligned} \mathcal{L} &= (\partial_\mu \phi + ieA_\mu \phi)^\dagger (\partial^\mu \phi + ieA^\mu \phi) + \bar{\psi}(i\not{\partial} - e\not{A})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \partial_\mu \phi^\dagger \partial^\mu \phi + ie \left[\partial_\mu \phi^\dagger A^\mu \phi - \partial_\mu \phi A^\mu \phi^\dagger \right] + e^2 \phi^\dagger \phi A_\mu A^\mu \\ &\quad + \bar{\psi}(i\gamma^\nu \partial_\nu)\psi - e\bar{\psi}\gamma^\rho A_\rho \psi \\ &\quad - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned}$$

For $\phi^\dagger$, we then have

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^\dagger)} = \partial_\mu (\partial^\mu \phi + ieA^\mu \phi) = \partial_\mu D^\mu \phi,$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi^\dagger} = -ie\partial_\mu \phi A^\mu + e^2 \phi A_\mu A^\mu = -ieA_\mu D^\mu \phi.$$

Thus ϕ satisfies the gauge-invariant (and massless) version of the Klein-Gordon equation:

$$D_\mu D^\mu \phi^\dagger = 0.$$

Meanwhile, for $\bar{\psi}$:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} = 0,$$

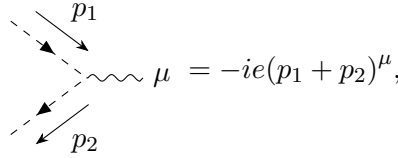
and

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = i\not{\partial}\psi - e\not{A}\psi = i\not{D}\psi.$$

Thus ψ satisfies a gauge-invariant and massless version of the Dirac equation:

$$i\not{D}\psi = 0.$$

b) This theory has three interaction vertices. One is given by

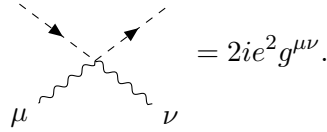


$$\mu = -ie(p_1 + p_2)^\mu, \quad (4)$$

where the dashed lines represent charged scalars. Draw the two other vertices, and give their order in e (you do not need to derive the full Feynman rules).

Solution: One is the standard photon-fermion interaction vertex from QED, with order e .

From writing out the covariant derivative explicitly, the Lagrangian contains a term $e^2 \phi^\dagger \phi A_\mu A^\mu$. This is an interaction term connecting two scalars and two photons, with order e^2 ; the vertex is then given by (the full Feynman rule is included just for completeness)



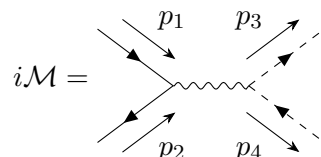
$$= 2ie^2 g^{\mu\nu}.$$

c) Draw the leading-order Feynman diagram for the process $\bar{\psi}\psi \mapsto \phi^\dagger\phi$ (i.e., a fermion-antifermion pair $\mapsto$ scalar boson particle-antiparticle pair). Show that the differential cross-section in the center-of-mass frame $\frac{d\sigma}{d\Omega_{\text{cm}}}$ for the process, to lowest order in e , is given by

$$\frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{\alpha^2}{8E_{\text{cm}}^2} \sin^2 \theta, \quad (5)$$

where $\alpha = e^2/(4\pi)$, E_{cm} is the center-of-mass energy, and θ is the angle between the incoming and outgoing particle directions in the center-of-mass frame.

Solution: The leading-order matrix element for the process is



$$i\mathcal{M} = \bar{v}^r(p_2)(-ie\gamma^\mu)u^s(p_1)\frac{-ig_{\mu\nu}}{(p_1 + p_2)^2}(-ie)(p_3 - p_4)^\nu,$$

where s and r represent the spin states of the incoming fermions. Averaging over these, we find for the spin-averaged squared matrix element

$$\begin{aligned}\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4(p_1 + p_2)^4} \text{tr}[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu] (p_3 - p_4)_\mu (p_3 - p_4)_\nu \\ &= \frac{e^4}{(p_1 + p_2)^2} [p_2^\mu p_1^\nu - p_1 \cdot p_2 g^{\mu\nu} + p_1^\mu p_2^\nu] (p_3 - p_4)_\mu (p_3 - p_4)_\nu \\ &= \frac{e^4}{(p_1 + p_2)^4} \left[2p_1 \cdot (p_3 - p_4) p_2 \cdot (p_3 - p_4) - p_1 \cdot p_2 (p_3 - p_4)^2 \right].\end{aligned}$$

To continue, we need to parametrize the particle momenta; this is most easily done in the center-of-mass frame, where we can write (since all particles are massless)

$$\begin{aligned}p_1 &= \frac{E_{\text{cm}}}{2} (1, 0, 0, 1), \\ p_2 &= \frac{E_{\text{cm}}}{2} (1, 0, 0, -1), \\ p_3 &= \frac{E_{\text{cm}}}{2} (1, \sin \theta, 0, \cos \theta), \\ p_4 &= \frac{E_{\text{cm}}}{2} (1, -\sin \theta, 0, -\cos \theta).\end{aligned}$$

Plugging this in leaves

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{2} \sin^2 \theta,$$

which leads to (using that the particles move at light speed, since they are massless) 11

$$\frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 E_{\text{cm}}^2} = \frac{\alpha^2}{8E_{\text{cm}}^2} \sin^2 \theta.$$

Problem 3

In this problem we consider a free, scalar, hermittian quantum field $\phi(x)$. The field satisfies the Klein–Gordon equation with mass m , and can be written in the usual form (see formula sheet).

a) Which of the following quantities must vanish due to causality?

1. the correlation function $\langle 0 | \phi(x) \phi(y) | 0 \rangle$ for any $x - y$;
2. the correlation function $\langle 0 | \phi(x) \phi(y) | 0 \rangle$ for spacelike $x - y$;
3. the correlation function $\langle 0 | \phi(x) \phi(y) | 0 \rangle$ for timelike $x - y$;
4. the commutator $[\phi(x), \phi(y)]$ for any $x - y$;
5. the commutator $[\phi(x), \phi(y)]$ for spacelike $x - y$;

6. the commutator $[\phi(x), \phi(y)]$ for timelike $x - y$;
7. the expectation value of $\phi(x)\phi(x)$ in a single particle state for all x ;
8. the expectation value of $\phi(x)\phi(x)$ in a single particle state for $\mathbf{x} \rightarrow \infty$ when the particle is located in the origin.

Solution: The only of these quantities vanishing due to causality is the commutator $[\phi(x), \phi(y)]$ for spacelike $x - y$. This is because a measurement of $\phi(x)$ should not influence a measurement of $\phi(y)$. For timelike $x - y$ there can be such an influence. The correlation function can be nonzero everywhere (due to vacuum fluctuations), and the expectation value of the squared field can be nonzero everywhere, both in the vacuum state and in a single particle state.

b) Show that the correlation function $D(x - y) \equiv \langle 0 | \phi(x) \phi(y) | 0 \rangle$ can be written in the form

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2E_{\mathbf{k}}} e^{-ik \cdot (x-y)}. \quad (6)$$

From now on, we consider the field at $t = 0$, and denote it $\phi(\mathbf{x})$. The field $\phi(\mathbf{x})$ itself is not a proper observable, since it is a so-called operator-valued distribution. This is seen e.g. from the commutator relation, involving a Dirac delta distribution. To obtain an observable ϕ_σ , we must therefore integrate (smear) the fields with a test function $f_\sigma(\mathbf{x})$:

$$\phi_\sigma \equiv \int d^3 x f_\sigma(\mathbf{x}) \phi(\mathbf{x}). \quad (7)$$

For ϕ_σ to represent the field close to $\mathbf{x} = 0$, this test function must be nonzero only in a small neighborhood of $\mathbf{x} = 0$. We let $f_\sigma(\mathbf{x})$ be 3D gaussian distribution

$$f_\sigma(\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}} e^{-\frac{|\mathbf{x}|^2}{2\sigma^2}}. \quad (8)$$

This function is normalized $\int d^3 x f_\sigma(\mathbf{x}) = 1$ and has Fourier transform

$$F_\sigma(\mathbf{k}) \equiv \int d^3 x f_\sigma(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} = e^{-\frac{\sigma^2 |\mathbf{k}|^2}{2}}. \quad (9)$$

c) Calculate the expectation value of the smeared field in the vacuum state, $\langle 0 | \phi_\sigma | 0 \rangle$.

Solution:

$$\langle 0 | \phi_\sigma | 0 \rangle = \int d^3 x f_\sigma(\mathbf{x}) \langle 0 | \phi(\mathbf{x}) | 0 \rangle = 0,$$

where we have used that the field only contains terms with creation or annihilation operators.

d) Find an expression for $\langle 0|\phi_\sigma^2|0\rangle$, and show that for the case $\sigma \gg m^{-1}$ it is roughly

$$\langle 0|\phi_\sigma^2|0\rangle = \frac{1}{16\pi^{\frac{3}{2}}m\sigma^3}. \quad (10)$$

Given: The Γ function is defined by

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t}, \quad (11)$$

with some values $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$, and recursive formula $\Gamma(z+1) = \Gamma(z)z$.

More generally, one can show (you don't have to do this):

$$\langle 0|\phi_\sigma^2|0\rangle = \begin{cases} \frac{1}{8\pi^2\sigma^2}, & \sigma \ll m^{-1}, \\ \frac{1}{16\pi^{\frac{3}{2}}m\sigma^3}, & \sigma \gg m^{-1}. \end{cases} \quad (12)$$

Solution: Using (7) and (6) we find

$$\langle 0|\phi_\sigma^2|0\rangle = \int d^3x \int d^3y f_\sigma(\mathbf{x}) f_\sigma(\mathbf{y}) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{\mathbf{k}}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

Taking the x and y integrals first, and identifying the Fourier transforms, we have

$$\langle 0|\phi_\sigma^2|0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{\mathbf{k}}} F_\sigma^2(\mathbf{k}) = \frac{1}{(2\pi)^2} \int_0^\infty dk \frac{k^2 e^{-\sigma^2 k^2}}{\sqrt{k^2 + m^2}}.$$

The function $e^{-\sigma^2 k^2}$ is negligible for $k \gtrsim \sigma^{-1}$. Thus for $\sigma^{-1} \ll m$, we can set $\sqrt{k^2 + m^2} = m$ in the integral. On the other hand, for $\sigma^{-1} \gg m$ we can set $\sqrt{k^2 + m^2} = k$ for most k 's contributing to the integral:

$$\langle 0|\phi_\sigma^2|0\rangle = \begin{cases} \frac{1}{(2\pi)^2} \int_0^\infty dk k e^{-\sigma^2 k^2}, & \sigma^{-1} \gg m \\ \frac{1}{(2\pi)^2 m} \int_0^\infty dk k^2 e^{-\sigma^2 k^2}, & \sigma^{-1} \ll m. \end{cases}$$

Performing the integral in these two special cases with the help of the Γ function, we find

$$\langle 0|\phi_\sigma^2|0\rangle = \begin{cases} \frac{1}{8\pi^2\sigma^2}, & \sigma \ll m^{-1}, \\ \frac{1}{16\pi^{\frac{3}{2}}m\sigma^3}, & \sigma \gg m^{-1}. \end{cases}$$

e) What is the physical significance of the expression $\langle 0|\phi_\sigma^2|0\rangle$? Explain what the result (12) means physically.

Solution: The expression

$$\langle 0|(\Delta\phi)^2|0\rangle \equiv \langle 0|\phi_\sigma^2|0\rangle - \langle 0|\phi_\sigma|0\rangle^2$$

is the expected vacuum fluctuations of the smeared field. Since $\langle 0|\phi_\sigma|0\rangle = 0$, we have $\langle 0|\phi_\sigma^2|0\rangle = \langle 0|(\Delta\phi)^2|0\rangle$, so $\langle 0|\phi_\sigma^2|0\rangle$ measures the vacuum fluctuations.

Recall that σ is the radius of the smearing function, i.e., the size of the region that we smear the field. Regardless of the behavior $1/\sigma^2$ or $1/\sigma^3$, we see that if we smear the field in a small region, the vacuum fluctuations become large, while if the region is large, the vacuum fluctuations become small. We can think of the smearing as an averaging which washes out the fluctuations.

Formulas

Lagrangian field theory:

$$\begin{aligned}
 S &= \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \\
 \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} &= 0 \\
 \pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\
 H &= \int d^3x (\pi \dot{\phi} - \mathcal{L})
 \end{aligned}$$

Noether:

$$\begin{aligned}
 \phi_r &\mapsto \phi_r + \alpha \Delta \phi_r \\
 \mathcal{L} &\mapsto \mathcal{L} + \alpha \partial_\mu J^\mu \\
 j^\mu &= \sum_r \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} \Delta \phi_r - J^\mu
 \end{aligned}$$

Lorentz transformations, boost along the z -axis:

$$\begin{aligned}
 t' &= \gamma(t + \beta z) \\
 z' &= \gamma(z + \beta t)
 \end{aligned}$$

Klein-Gordon field:

$$\begin{aligned}
 \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right) \\
 (\partial^2 + m^2)\phi &= 0 \\
 [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}).
 \end{aligned}$$

Dirac field:

$$\begin{aligned}
 \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_r \left(a_{\mathbf{p}}^r u^r(\mathbf{p}) e^{-ipx} + b_{\mathbf{p}}^{r\dagger} v^r(\mathbf{p}) e^{ipx} \right) \\
 (i\not{\partial} - m)\psi &= 0 \\
 \{a_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}\} &= \{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs} \\
 u^s(p) &= \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix}, \quad v^s(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{bmatrix} \\
 \bar{u}^r(p) u^s(p) &= -\bar{v}^r(p) v^s(p) = 2m \delta^{rs} \\
 \not{p} u(p) &= m u(p), \quad \bar{u}(p) \not{p} = \bar{u}(p) m, \\
 \not{p} v(p) &= -m v(p), \quad \bar{v}(p) \not{p} = -\bar{v}(p) m \\
 \sum_{s=1}^2 u^s(p) \bar{u}^s(p) &= \not{p} + m, \quad \sum_{s=1}^2 v^s(p) \bar{v}^s(p) = \not{p} - m
 \end{aligned}$$

Chiral representation:

$$\begin{aligned}
 \gamma^0 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}; \\
 \sigma^1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
 \end{aligned}$$

Definitions:

$$\begin{aligned} f \overset{\leftrightarrow}{\partial}_0 g &= f \partial_0 g - (\partial_0 f) g \\ \sigma^\mu &\equiv (1, \sigma^i), \quad \bar{\sigma}^\mu \equiv (1, -\sigma^i) \\ \not{a} &\equiv \gamma^\mu a_\mu \text{ (4-vector } a^\mu) \end{aligned}$$

Identities:

$$\begin{aligned} \sigma^i \sigma^j &= \delta^{ij} + i\epsilon^{ijk} \sigma^k \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\ \gamma^5 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \\ \gamma^\mu \gamma_\mu &= 4 \\ \gamma^\mu \gamma^\nu \gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma_\mu &= 4g^{\nu\alpha} \\ \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\rho \gamma_\mu &= -2\gamma^\rho \gamma^\alpha \gamma^\nu \\ \text{tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu} \\ \text{tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\rho} g^{\nu\sigma} + 4g^{\mu\sigma} g^{\nu\rho} \\ \text{tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] &= 0, \text{ for } n \text{ odd} \end{aligned}$$

$(\bar{v}^s(p) \Gamma u^r(k))^\dagger = \bar{u}^r(k) \Gamma' v^s(p)$, Gamma matrices in Γ' reversed compared to that in Γ .

Ward's identity:

Let $i\mathcal{M}_\mu(q)$ be a diagram where a photon line with four-momentum q at vertex μ is removed.

Then $\mathcal{M}_\mu(q)q^\mu = 0$.

Gordon's identity:

$$\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p), \text{ where } \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \text{ and } q^\mu = p'^\mu - p^\mu.$$

Cross-sections and decay rates:

$$\begin{aligned} d\sigma &= \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_A, p_B \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A + p_B - \sum p_f) \\ d\Gamma &= \frac{1}{2m_A} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\quad \times |\mathcal{M}(p_A = (m_A, \mathbf{0}) \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4(p_A - \sum p_f) \end{aligned}$$

Two-particle phase space:

$$\int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2) = \int d\Omega_{\text{cm}} \frac{1}{16\pi^2} \frac{|\mathbf{p}_1|}{E_{\text{cm}}},$$

$E_{\text{cm}} = E_1 + E_2$ in the center-of-mass frame; Ω_{cm} = solid angle for particle

1. For decay of a single particle, set $p_B = 0$.

Lagrangians and Feynman rules

The following rules are used to obtain the amplitude $i\mathcal{M}$, and given in momentum space. Dots represent vertices in all of the diagrams listed below. For consistency between ϕ^4 theory and Yukawa theory, scalar particles are represented by dashed lines, and their masses by M , in both cases.

Common for all theories:

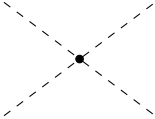
- Impose momentum conservation at each vertex
- Integrate over each undetermined momentum, $\int \frac{d^4k}{(2\pi)^4}$

 ϕ^4 theory

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}M^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

• Propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

• External line: $\xrightarrow{p} \bullet = 1$

• Vertex:  $= -i\lambda$

- Divide by the symmetry factor

Yukawa theory:

$$\mathcal{L}_Y = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\psi\phi$$

• Scalar propagator: $\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - M^2 + i\epsilon}$

• Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

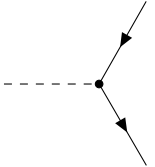
• External scalar: $\xrightarrow{p} \bullet = 1$

• Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$

• Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$

• Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$

• Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$

• Vertex:  $= -ig$

- Work out the overall sign of the diagram
- For closed fermion loops, include a factor (-1) and a trace.

QED:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad e = -|e|$$

- Photon propagator: $\mu \xrightarrow{p} \nu = \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}$
- Fermion propagator: $\bullet \xrightarrow{p} \bullet = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$
- Incoming photon: $\mu \xrightarrow{p} = \epsilon_\mu(p)$
- Outgoing photon: $\bullet \xrightarrow{p} \mu = \epsilon_\mu^*(p)$
- Incoming fermion (spin s): $\xrightarrow{p} \bullet = u^s(p)$
- Incoming antifermion (spin s): $\xleftarrow{p} \bullet = \bar{v}^s(p)$
- Outgoing fermion (spin s): $\bullet \xrightarrow{p} = \bar{u}^s(p)$
- Outgoing antifermion (spin s): $\bullet \xleftarrow{p} = v^s(p)$
- Vertex: $\mu \xrightarrow{\quad} \bullet = -ie\gamma^\mu$
- Work out the overall sign of the diagram
- For closed fermion loops, include a factor (-1) and a trace.