

Solutions [and grading guidelines]

for the midterm exam in FYS5120

[General grading guidelines: Stated points are given for arriving at the respective expression in a fully satisfactory way. Whole point subtractions for bad logic, physically wrong statements or expressions that are *physically* wrong (e.g. wrong dimensions, energy not conserved etc.)! ‘Obvious’ small math errors (e.g. wrong prefactors) only 0.5 pt. When possible, no subtraction for follow-up mistakes. Up to 1 extra point for explanations demonstrating special insight (but this cannot result in more points than available for a given problem).]

Problem 1

a) The generating functional for QED is defined as

$$\begin{aligned} Z[J, \eta, \bar{\eta}] &\equiv \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[i \int d^4x (\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} + J_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta) \right] \\ &= \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[i \int d^4x (\mathcal{L}_{\text{free}} + J_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta) \right] \left(1 + i \int d^4x' \mathcal{L}_{\text{int}} + \dots \right), \end{aligned} \quad (1)$$

where $\mathcal{L}_{\text{free}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\cancel{\partial} - m)\psi$ and $\mathcal{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi A_\mu$ [1 pt]. Using the basic properties of the functional derivative, we can hence re-write this as

$$Z[J, \eta, \bar{\eta}] = \left(1 - ie \int d^4x' \left(i \frac{\delta}{\delta\eta(x')} \right) \gamma^\mu \left(-i \frac{\delta}{\delta\bar{\eta}(x')} \right) \left(-i \frac{\delta}{\delta J_\mu(x')} \right) + \dots \right) Z_0 \quad (2)$$

$$\equiv \left(1 - e \frac{\delta}{\delta\eta_{x'}} \gamma_\mu \frac{\delta}{\delta\bar{\eta}_{x'}} \frac{\delta}{\delta J_{\mu,x'}} + \dots \right) Z_0, \quad \text{[1 pt]} \quad (3)$$

where (using the results provided in the exam appendix)

$$Z_0 \equiv \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[i \int d^4x (\mathcal{L}_{\text{free}} + J_\mu a^\mu + \bar{\eta}\psi + \bar{\psi}\eta) \right] \quad (4)$$

$$\propto \exp \left[-\frac{1}{2} \int d^4x d^4y J_\mu(y) D_F^{\mu\nu}(y-x) J_\nu(x) \right] \exp \left[-\int d^4x d^4y \bar{\eta}(y) S_F(y-x) \eta(x) \right]$$

$$\equiv \exp \left[-\frac{1}{2} J_{\mu,y} D_{yx}^{\mu\nu} J_{\mu,y} - \bar{\eta}_y S_{yx} \eta_x \right]. \quad (5)$$

The QED vertex is determined by the 3-point function

$$\langle \Omega | A^\mu(x_1) \psi(x_2) \bar{\psi}(x_3) | \Omega \rangle = -i \frac{1}{Z[0,0,0]} \frac{\delta}{\delta J_{\mu,x_1}} \frac{\delta}{\delta \bar{\eta}_{x_2}} \frac{\delta}{\delta \eta_{x_3}} Z[0,0,0] \quad (6)$$

$$\begin{aligned} &= -i \frac{\frac{\delta}{\delta J_{\mu,x_1}} \frac{\delta}{\delta \bar{\eta}_{x_2}} \frac{\delta}{\delta \eta_{x_3}} \left(1 - e \frac{\delta}{\delta \eta_{x'}} \gamma_\nu \frac{\delta}{\delta \bar{\eta}_{x'}} \frac{\delta}{\delta J_{\nu,x'}} + \dots \right) Z_0[0,0,0]}{\left(1 - e \frac{\delta}{\delta \eta_{x'}} \gamma_\nu \frac{\delta}{\delta \bar{\eta}_{x'}} \frac{\delta}{\delta J_{\nu,x'}} + \dots \right) Z_0[0,0,0]} \\ &= ie \frac{\delta}{\delta J_{\mu,x_1}} \frac{\delta}{\delta \bar{\eta}_{x_2}} \frac{\delta}{\delta \eta_{x_3}} \frac{\delta}{\delta \eta_{x'}} \gamma_\nu \frac{\delta}{\delta \bar{\eta}_{x'}} \frac{\delta}{\delta J_{\nu,x'}} \quad (7) \end{aligned}$$

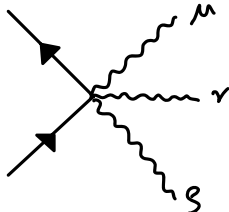
$$\exp \left[-\frac{1}{2} J_{\rho,y} D_{yx}^{\rho\sigma} J_{\sigma,x} - \bar{\eta}_y S_{yx} \eta_x \right]_{J=\bar{\eta}=\eta=0} + \dots \text{[1 pt]}$$

So far, we have only kept leading order contributions in e , but not discarded any terms otherwise. We now restrict ourselves to the specific contractions that lead to the vertex rule (i.e. all external points connected to the same internal point x'):

$$\begin{aligned} \langle \Omega | A^\mu(x_1) \psi(x_2) \bar{\psi}(x_3) | \Omega \rangle &\supset i e \frac{\delta}{\delta J_{\mu, x_1}} \frac{\delta}{\delta \bar{\eta}_{x_2}} \frac{\delta}{\delta \eta_{x_3}} (+\bar{\eta}_u S_{ux'}) \gamma_\nu (-S_{x'v} \eta_v) (-J_{\rho, w} D_{wx'}^{\rho\nu}) \Big|_{J=\bar{\eta}=\eta=0} \\ &= -ie S_{x_2 x'} \gamma_\nu S_{x' x_3} D_{x_1 x'}^{\mu\nu}. \end{aligned} \quad (8)$$

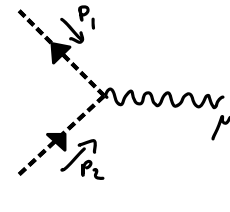
Note that in each of the above steps care must be taken with respect to signs resulting from commuting Grassmann variables or their derivatives. Truncating the external legs represented by the propagators S and D in the final expression, as dictated by the LSZ formula, finally leaves the familiar QED vertex rule $-ie\gamma^\mu$ **[1 pt; only -0.5 for at most one sign error]**.

b) Since the mass dimension of $\mathcal{L}$ must be 4, we have $[c_1] = -2$, $[c_2] = 0$ and $[c_3] = -1$ **[0.5 pt each]**. Assuming all particles to be ingoing, the vertex rules are given by: **[0.5 pt per correct diagram, including arrow directions; 1 pt per correct rule, -0.5 when not fully simplified]**

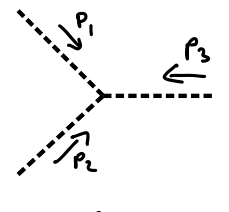


$$= -2i c_1 \gamma^5$$

$$\times \{ \gamma^\mu \gamma^{s\nu} + \gamma^s \gamma^{\mu\nu} + \gamma^\nu \gamma^{s\mu} \}$$



$$= i c_2 (p_1 - p_2)^\mu$$



$$= +i c_3 \sum_{i=1}^3 \sum_{j \neq i} p_i \cdot p_j$$

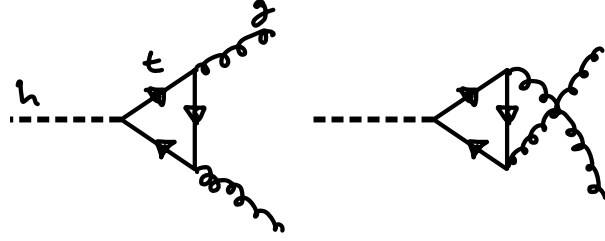
$$= +i 2 c_3 (p_3^2 + p_1 \cdot p_2)$$

Note i) a symmetry factor of $n!$ for any field appearing n times, ii) the replacement of $\partial_\mu \rightarrow -ip_\mu$ for ingoing momenta, iii) assigning particles and antiparticles correctly, iv) using integration by parts/ momentum conservation for simplification.

Problem 2

a) The Higgs field H must transform in the same representation as the fermions, i.e. in the fundamental representation of $SU(3)$; this is a complex triplet, consisting of six real components **[1 pt]**. The dimension of the adjoint representation of $SU(N)$ is $N^2 - 1$. The number of spontaneously broken symmetries that is hence $(3^2 - 1) - (2^2 - 1) = 5$, i.e. five of the gauge bosons are massive **[1 pt]**. This leaves $8 + 3 + 1 - 5 = 7$ massless gauge bosons from the original gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ **[0.5 pt]**. Out of the six Higgs triplet d.o.f., five must hence be Goldstone modes – leaving still only one physical Higgs boson **[1 pt]**. Degrees of freedom before SSB: 12 massless gauge bosons + 1 complex triplet scalar = $12 \times 2 + 6 = 30$ **[1 pt]**. Degrees of freedom after SSB: 7 massless gauge bosons + 5 massive gauge bosons + 1 real scalar = $7 \times 2 + 5 \times 3 + 1 = 30$ **[0.5 pt]**.

b) The two diagrams are the top-loops depicted to the right **[0.5 pt each]**. The diagrams must be loop-induced since gluons only couple to quarks. Top quarks, furthermore, have the (by far) largest Yukawa coupling in the SM **[1 pt for any valid explanation]**.



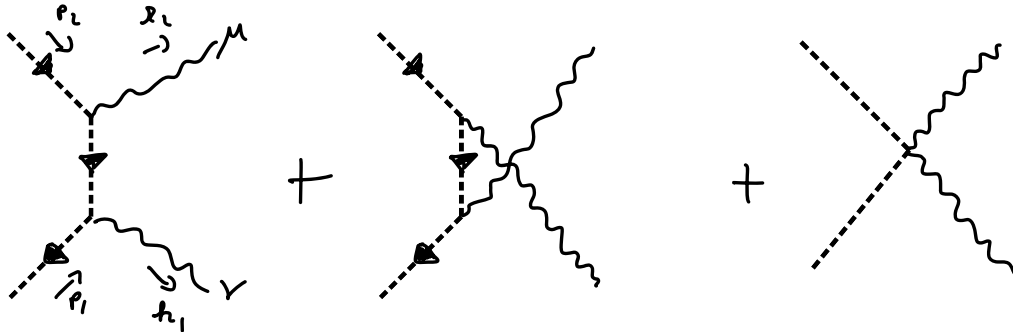
The same type of triangle diagram will contribute to $h \rightarrow \gamma\gamma$, with $\mathcal{M} \propto Q_f^2 y_f$ for *any* SM fermion f with charge Q_f (i.e. not only quarks). Again, the top loop diagram will give the by far largest value of $Q_f^2 y_f$ for all f , more than compensating for example the larger value of Q for leptons compared to quarks **[1 pt]**. With photon final states, the loop particles must be charged; this leaves only $W^\pm$ loops as further potential candidates **[1 pt]**. There are two triangle diagrams (literally replacing the fermions with $W^\pm$) and one further diagram involving a W loop and a 4-point coupling $W^+W^-\gamma\gamma$. Each of these has the same coupling structure, with an $SU(2)$ factor of g coming from the hW^+W^- vertex and a factor of α_{em} from the $W - \gamma$ couplings (since $Q_W^2 = 1$). The relative ratio of the two contributions to the rates should thus naively be expected to be given by roughly

$$\frac{|\mathcal{M}|_W^2}{|\mathcal{M}|_t^2} \sim \frac{g^2}{y_t^2 (2/3)^4} = \frac{(2m_W/v)^2}{(\sqrt{2}m_t/v)^2 (2/3)^4} = \frac{81}{8} \frac{m_W^2}{m_t^2} \sim 2, \quad (9)$$

using $m_t \sim 170 \text{ GeV}$ and $m_W \sim 80 \text{ GeV}$ **[1 pt for obtaining the correct order of magnitude by using similar reasoning]**. The actual ratio is larger by almost an order of magnitude – even though we neglected a color factor 3^2 pulling in the opposite direction. However, dimensional analysis tells us that the top loop amplitude – with dimension-less couplings – should scale as $1/m_t$; The W loop on the other hand has derivative couplings; with $\partial_\mu \sim p_\mu \sim m_h$ being the typical energy scale, we hence expect a scaling m_h^3/m_W^4 . At the level of the rates, this amounts to an enhancement of $(m_t m_h^3/m_W^4)^2 \sim 70$ – which brings our very naive estimate of the ratio to an agreement of better than a factor of two compared to a complicated loop calculation. **[Up to 1 bonus point for reasoning including effects in these directions.]**

Problem 3

a) The relevant diagrams (with momentum and Lorentz index assignments) are



[0.5 pts per diagram] The relevant vertex rules come from the $|D_\mu \Phi|^2$ term; in particular, $\mathcal{L} \supset igB^\mu(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*)$ determines the 3-point coupling (with $\partial_\mu \rightarrow -ip_\mu$ for ingoing momenta) and $\mathcal{L} \supset g^2 \eta_{\mu\nu} \Phi \Phi^* B^\mu B^\nu$ the 4-point coupling. The amplitude thus becomes $\mathcal{M} = \mathcal{M}_{tu} + \mathcal{M}_s$, with

$$\mathcal{M}_{tu} = i(ig)^2 \epsilon_\mu^*(k_2) \epsilon_\nu^*(k_1) \left[\frac{(2p_2 - k_2)^\mu (-2p_1 + k_1)^\nu}{t - m_\Phi^2} + \frac{(2p_2 - k_1)^\nu (-2p_1 + k_2)^\mu}{u - m_\Phi^2} \right] \quad [1 \text{ pt}] \quad (10)$$

$$= -2ig^2 \epsilon_\mu^*(k_2) \epsilon_\nu^*(k_1) \left[\frac{p_2^\mu p_1^\nu}{p_1 \cdot k_1} + \frac{p_2^\nu p_1^\mu}{p_2 \cdot k_1} \right] \quad [1 \text{ pt}] \quad (11)$$

$$\mathcal{M}_s = 2ig^2 \eta^{\mu\nu} \epsilon_\mu^*(k_2) \epsilon_\nu^*(k_1) = 2ig^2 \epsilon^*(k_2) \cdot \epsilon^*(k_1), \quad [1 \text{ pt}] \quad (12)$$

where we used $k_i \cdot \epsilon(k_i) = 0$, $t = (p_1 - k_1)^2$, $u = (p_2 - k_1)^2$, $p_i^2 = m_\Phi^2$ and $k_i^2 = 0$.

It suffices to show the Ward identity for one of the final-state momenta; we choose $\epsilon_\nu^*(k_1) \rightarrow k_{1\nu}$ for simplicity, given the form of our result above. We then find

$$\mathcal{M}_s \rightarrow 2ig^2 \epsilon^*(k_2) \cdot k_1, \quad [0.5 \text{ pt}] \quad (13)$$

$$\mathcal{M}_{tu} \rightarrow -2ig^2 \epsilon_\mu^*(k_2) k_{1\nu} \left[\frac{p_2^\mu p_1^\nu}{p_1 \cdot k_1} + \frac{p_2^\nu p_1^\mu}{p_2 \cdot k_1} \right] \quad (14)$$

$$= -2ig^2 \eta^{\mu\nu} \epsilon^*(k_2) \cdot (p_2 + p_1) = -2ig^2 \epsilon^*(k_2) \cdot k_1, \quad [1 \text{ pt}] \quad (15)$$

where we used momentum conservation along with $k_2 \cdot \epsilon(k_2) = 0$ in the final step. Hence, $\mathcal{M} = \mathcal{M}_{tu} + \mathcal{M}_s = 0$, as expected.

b) When computing $\sum_\epsilon |\mathcal{M}|^2$, the two physical transverse degrees of freedom, with $\epsilon(k_i) \cdot \mathbf{k}_i = 0$, should each give non-vanishing contributions to the sum **[0.5 pt]**; the *sum* of all contributions involving any of the other two polarizations, on the other hand, must vanish as these constitute unphysical degrees of freedom **[0.5 pt]**. For Abelian theories, the latter is automatically guaranteed by the Ward identity; for non-Abelian theories, this is *not* the case, leading to the necessity of ghosts to cancel the contributions from these unphysical degrees of freedom in loop diagrams **[1 pt]** (where they appear in propagators since $\eta^{\mu\nu} = |\epsilon\rangle \langle \epsilon|_{\mu\nu}$ effectively implements a sum over *all*, also unphysical, polarization states).

c) The unbroken Lagrangian is invariant under

$$\Phi \rightarrow e^{i\alpha} \Phi, \quad [0.5 \text{ pt}] \quad (16)$$

$$B_\mu \rightarrow B_\mu + g^{-1} \partial_\mu \alpha \quad [0.5 \text{ pt}] \quad (17)$$

for arbitrary space-time functions $\alpha(x)$. Taking advantage of $V(\Phi) = V(|\Phi|)$, a useful parametrization of the vev in terms of two real numbers v and β is $\langle \Phi \rangle = v e^{i\beta}$. Correspondingly, we make the ansatz

$$\phi(x) = (v + \sigma(x)/S) e^{i\beta + i\pi(x)/F}, \quad [1 \text{ pt}] \quad (18)$$

which satisfies as desired the requirement $\langle \sigma \rangle = 0$ for all vacuum states; here, S (F) is a constant of mass dimension zero (one). Plugging this into the original Lagrangian

gives

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu - igB_\mu)(v + \sigma/S)e^{i\beta+i\pi/F})|^2 - V(v + \sigma/S) \quad (19)$$

$$= -\frac{1}{4}F_{\mu\nu}^2 + \left| \frac{1}{S}\partial_\mu\sigma + \left(\frac{i}{F}\partial_\mu\pi - igB_\mu \right) (v + \sigma/S) \right|^2 - V(v + \sigma/S) \quad (20)$$

$$= -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}\left(\partial_\mu\pi - \sqrt{2}vgB_\mu\right)^2 \left(1 + \frac{\sigma}{\sqrt{2}v}\right)^2 - V\left(v + \frac{\sigma}{\sqrt{2}}\right), \quad (21)$$

where we have chosen $S = \sqrt{2}$ and $F = \sqrt{2}v$ in the last step in order to make the kinetic terms of σ and π canonically normalized **[2 pts for full result]**. This Lagrangian is invariant under

$$B_\mu \rightarrow B_\mu + g^{-1}\partial_\mu\alpha \quad \textbf{[0.5 pt]} \quad (22)$$

$$\pi \rightarrow \pi + \sqrt{2}v\alpha, \quad \textbf{[0.5 pt]} \quad (23)$$

for arbitrary space-time functions $\alpha(x)$. Note that we have a fully gauge-invariant theory – despite the fact that the gauge boson has obtained a mass $m_B = \sqrt{2}gv$.

d) In unitary gauge the Goldstone boson π disappears and there is only one diagram that contributes, coming from the four-point coupling $\mathcal{L} \supset \frac{1}{2}g^2B^2\sigma^2$ **[1 pt]**. We can hence immediately write down the amplitude for $\sigma\sigma \rightarrow BB$ as

$$i\mathcal{M} = 2ig^2\epsilon^*(k_1) \cdot \epsilon^*(k_2). \quad \textbf{[1 pt]} \quad (24)$$

Summing over all final-state polarizations in the usual way, we thus find **[1 pt]**

$$|\mathcal{M}|^2 = 4g^2 \left(-\eta_{\mu\mu'} + \frac{k_{1\mu}k_{1\mu'}}{m_B^2} \right) \left(-\eta^{\mu\mu'} + \frac{k_2^\mu k_2^{\mu'}}{m_B^2} \right) = 4g^2 \left(4 - 2 + \frac{(k_1 \cdot k_2)^2}{m_B^4} \right). \quad (25)$$

Instead, we can consider the three physical polarizations individually. Starting with two transverse directions

$$\epsilon_{1,2}^T(k) = (0, \hat{\mathbf{k}}_{1,2}^T) \quad \text{with } \hat{\mathbf{k}}_{1,2}^T \cdot \mathbf{k} = 0, \quad \hat{\mathbf{k}}_i^T \cdot \hat{\mathbf{k}}_j^T = \delta_{ij}, \quad (26)$$

and hence

$$i\mathcal{M}_1 = i\mathcal{M}_2 = -2ig^2. \quad \textbf{[1 pt]} \quad (27)$$

Writing $k_1 = (\omega, \mathbf{k})$, the longitudinal polarization takes the form

$$\epsilon_L^\mu(k_1) = \Lambda^\mu{}_\nu(0, 0, 0, 1)^\nu = \gamma(\beta, \hat{\mathbf{k}})^\mu = (k, \omega\hat{\mathbf{k}})/m_B, \quad \textbf{[0.5 pt]} \quad (28)$$

where $k \equiv |\mathbf{k}|$ and $\hat{\mathbf{k}} \equiv \mathbf{k}/k$. In the CMS frame, we have $k_2 = (\omega, -\mathbf{k})$ **[0.5 pt]** and hence $\epsilon_L^\mu(k_2) = (k, -\omega\hat{\mathbf{k}})/m_B$. Contracting therefore gives

$$i\mathcal{M}_L = 2ig(k^2 + \omega^2)/m_B^2 = 2ig^2 \frac{k_1 \cdot k_2}{m_B^2}. \quad \textbf{[1 pt]} \quad (29)$$

Both final states must have the same polarization, because the polarization vectors are orthonormal. Summing thus gives $\sum_{\epsilon} |\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + |\mathcal{M}_L|^2$, which evaluates to the the same as in Eq. (25). Note in particular, that just using $\sum \epsilon^\mu(k)^* \epsilon^\nu(k)^* = -\eta^{\mu\nu}$ does *not* correspond to removing the longitudinal degrees of freedom.

If we had chosen another gauge, both Goldstone and ghosts would (re-)appear in the theory. The former mix with the longitudinal d.o.f. of the gauge bosons, in particular, so the computation of $\mathcal{M}_L$ would give a different result. Correspondingly, one would have to modify the spin-sum, i.e. $\sum \epsilon^\mu(k)^* \epsilon^\nu(k)^* \neq -\eta^{\mu\nu} + k^\mu k^\nu / m_B^2$ **[1 pt]**, in order to retain the physically correct result.

e) In d) we found $\mathcal{M}_L = 2igm_B^{-2}(k_1 \cdot k_2) = igm_B^{-2}(s - m_B^2) \rightarrow igs/m_B^2$ **[0.5 pt]**. Such unphysical behaviour points to the existence of missing diagrams that would cancel the unwanted terms **[1 pt]**. In fact, we have so far not taken into account the σ self-couplings that will necessarily arise from the potential term, $\mathcal{L} \supset V(v + \sigma/\sqrt{2})$, which will necessarily (because of $v \neq 0$) contain a cubic term $\mathcal{L} \supset \mu\sigma^3$ **[0.5 pt]**. However, this only creates an additional s -channel diagram $\mathcal{M}_s \propto (s - m_\sigma)^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2)$ – which tends towards a constant for large s (and longitudinal modes), and hence cannot cancel the critical behaviour **[1 pt]**.

The solution consists instead in taking account t - and u -channel diagrams with vector boson propagators. While one might naively think that the two three-point vertices from $\mathcal{L} \supset \sqrt{2}g^2 v \sigma B^2$ imply a contribution at order g^4 , this is actually not true for the longitudinal components of the propagator because $m_B = \sqrt{sv}g$ **[1 pt]**. Concretely, the high- s limit of these two diagrams is given by

$$\begin{aligned} \mathcal{M}_{t+u}^{s \rightarrow \infty} &\rightarrow i^3 [2\sqrt{2}g^2 v]^2 \left(\frac{(p_1 - k_1)^\mu (p_1 - k_1)^\nu}{(t - m_B^2)m_B^2} + \frac{(p_2 - k_1)^\mu (p_2 - k_1)^\nu}{(u - m_B^2)m_B^2} \right) \epsilon_\mu^*(k_2) \epsilon_\nu^*(k_1) \\ &\rightarrow -2ig^2 \left(\frac{p_2^\mu p_1^\nu}{p_1 \cdot k_1} + \frac{p_1^\mu p_2^\nu}{p_2 \cdot k_1} \right) \epsilon_\mu^*(k_2) \epsilon_\nu^*(k_1), \text{ **[1 pt]** } \end{aligned} \quad (30)$$

where in the second step we set $m_\sigma \rightarrow 0$ and used the same type of simplifications as in a). In the high-energy limit, furthermore, we have $\epsilon_\mu^L(k) \rightarrow k^\mu/m_B$, and hence

$$-\mathcal{M}_{t+u}^{L,s \rightarrow \infty} \rightarrow 2ig^2(p_2 \cdot k_2 + p_1 \cdot k_2)/m_B^2 = 2ig^2 k_2 \cdot (k_1 + k_2)/m_B^2 \rightarrow 2ig^2 k_1 \cdot k_2/m_B^2, \quad (31)$$

thus exactly cancelling the unphysical $\mathcal{M} \propto s$ behaviour **[Yet another bonus point for actually demonstrating the cancellation]**.