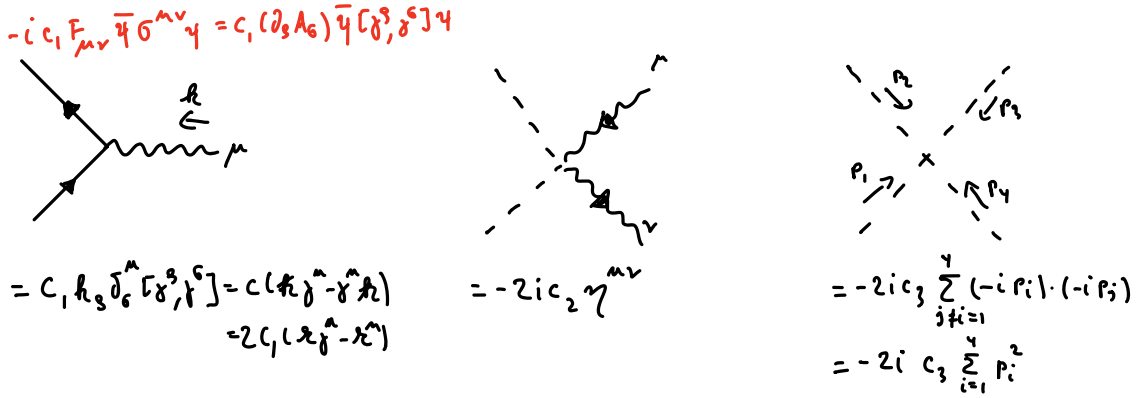
Solutions [and grading guidelines] for the midterm exam in FYS5120

[General grading guidelines: Stated points are given for arriving at the respective expression in a fully satisfactory way. Whole point subtractions for bad logic, physically wrong statements or expressions that are *physically* wrong (e.g. wrong dimensions, energy not conserved etc.)! 'Obvious' small math errors (e.g. wrong prefactors) only 0.5 pt. When possible, no subtraction for follow-up mistakes. Up to 1 extra point for explanations demonstrating special insight (but this cannot result in more points than available for a given problem).]

Problem 1

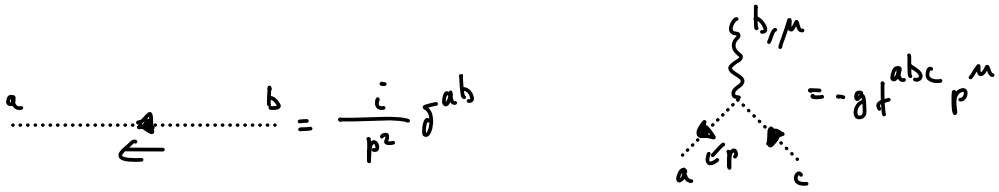
a) Since the mass dimension of $\mathcal{L}$ must be 4, we have $[c_1] = -1$, $[c_2] = 0$ and $[c_3] = -2$ **[0.5 pt each]**. Assuming all particles to be ingoing, the vertex rules are given by: **[0.5 pt per correct diagram, including arrow directions; 1 pt per correct rule, -0.5 when not fully simplified]**



Note i) a symmetry factor of $n!$ for any field appearing n times, ii) the replacement of $\partial_\mu \to -ip_\mu$ for ingoing momenta, iii) assigning particles and antiparticles correctly, iv), arrows *must* be included for charged field, v) using integration by parts/ momentum conservation for simplification.

(The final step in c) follows from $0 = (\sum_i p_i^\mu)^2 = \sum_i p_i^2 + \sum_{i\neq j} p_i \cdot p_j$, and noting that the momenta are not necessarily on-shell.

b) Ghosts live in the adjoint representation **[0.5 pt]**, so the covariant derivative is given by $D_\mu^{ab} = \delta^{ab}\partial_\mu - igA_\mu^c(t_G^c)_{ab} = \delta^{ab}\partial_\mu - gf^{cab}A_\mu^c$ The first term leads to $\mathcal{L} \supset -\bar{c}^a\Box c^a$, and thus to a standard propagator for a charged scalar as shown below **[0.5 pt for diagram, 1 pt for fully correct rule]**. The second term can be re-written as $\mathcal{L} \supset gf^{cab}\bar{c}^a\partial^\mu(A_\mu^c c^b) = -gf^{abc}(\partial^\mu\bar{c})A_\mu^c c^b$ **[0.5 pt]**, leading to **[0.5 pt for diagram, 1 pt for fully correct rule]**

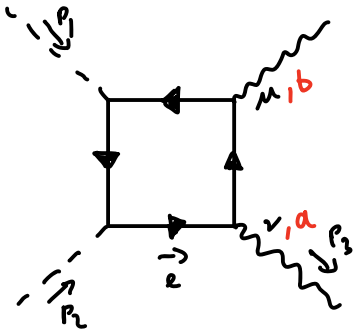


Problem 2

a) The gauge group of the standard model is given by $G = U(1)_Y \times SU(2)_L \times SU(3)_c$ **[0.5 pt]**. As gauge bosons transform in the adjoint representation, which has $N^2 - 1$ generators for $SU(N)$ **[0.5 pt]**, this implies that there are 8 gauge bosons for $SU(3)$ (the gluons), 3 electroweak gauge bosons for $SU(2)$ and the weak hypercharge boson. Gauge symmetry forbids mass terms, so each of these contributes with 2 physical d.o.f., i.e. 24 **[0.5 pt]** in total. The Higgs field is by construction only charged under $U(1)_Y \times SU(2)$, and must transform in the same representation as the fermions. This constrains it to be a complex doublet, thus adding further 4 (real) **[0.5 pt]** degrees of freedom.

When the Higgs field acquires a vacuum expectation value, it *breaks the original symmetry* as $U(1)_Y \times SU(2)_L \to U(1)_{\rm em}$ **[0.5pt]**. Three of the original gauge bosons thus acquire a mass by 'eating' the corresponding components of the Higgs doublet (those are the observed $W^\pm$ and Z bosons, with 3 real degrees of freedom each) **[0.5 pt]**. The remaining massless vector boson is the photon, and the remaining d.o.f. of the Higgs doublet is the physical Higgs field; the gluons are unaffected and remain massless **[0.5 pt]**. Both before and after SSB, the total number of bosonic, real d.o.f. in the SM is thus $1 \times 2 + 3 \times 2 + 8 \times 2 + 4 = 1 \times 2 + 3 \times 3 + 8 \times 2 + 1 = 28$. **[0.5 pt]**

b) The annihilation $\chi\chi \to \gamma\gamma$ can only happen via top box diagrams like the one shown in the figure **[1 pt]**. Starting from an arbitrary vertex, there are 3! possibilities to connect the other external legs to the top loop – so there are in total 6 different diagrams **[1 pt]**. For gluon emission there is an additional topology with a top triangle and an s-channel gluon; there are two such diagrams because of the two ways the incoming χ can be attached to the top loop **[1 pt]**.



Including relevant couplings, the fermion loop has the structure

$$\mathcal{M} \propto A_{\rm photon} \equiv 3Q_t^2 e^2 {\rm Tr}[S_F(l)(\lambda+\lambda_5\gamma^5)S_F(l-p_2)(\lambda+\lambda_5\gamma^5)S_F(l-p_1-p_2)\gamma^\mu S_F(l-p_3)\gamma^\nu] \tag{1}$$

in the case of photons, with S_F the top propagator and l the loop momentum (see also figure). $Q_t = 2/3$ is the charge of the top quark and the factor 3 accounts for the fact that there are three top colors that must be summed over in the loop. This is replaced by

$$\mathcal{M} \propto A_{\rm gluon} \equiv g^2 {\rm Tr}[S_F(l)(\lambda+\lambda_5\gamma^5)S_F(l-p_2)(\lambda+\lambda_5\gamma^5)S_F(l-p_1-p_2)\gamma^\mu t^b S_F(l-p_3)\gamma^\nu t^a] \tag{2}$$

in the case of gluon emission (red labels in the figure), where the t^a are the $SU(3)$ generators in the fundamental representation **[0.5 pt each for i) correct placement of t^a, ii) $e \to g$, iii) the factor Q_t^2 and iv) the factor 3]**. Note that the matrix elements are identical otherwise (apart from the color index of the gluon polarization vectors). Since the quark propagators are diagonal in color space, we can hence simplify further:

$$\frac{3Q_t^2\alpha_{\rm em}}{\alpha_s}\frac{A_{\rm gluon}}{A_{\rm photon}} = {\rm tr}[t^b t^a] = \delta^{ab} \times C(r) = \frac{1}{2}\delta^{ab} \textbf{ [1 pt]}. \tag{3}$$

Here, the lower case 'tr' indicates a trace only in color space (which gave a factor of 3 in the photon case, acting on the unit matrix). Note that this ratio is the same for each individual diagram, hence it remains the same independently of the actual number of diagrams. Squaring, and summing over all final states thus gives

$$\sum |\mathcal{M}_{\text{gluon}}|^2 = \frac{\alpha_s^2}{9Q_t^4\alpha_{\text{em}}^2} \sum |\mathcal{M}_{\text{photon}}|^2 \times \frac{1}{4} \underbrace{\delta^{ab}\delta^{ab}}_{=\delta^{aa}=d(G)=8} \,. \tag{4}$$

In total, we therefore find $\sigma_{\chi\chi\to gg}/\sigma_{\chi\chi\to\gamma\gamma} = \frac{9}{8}\alpha_s^2/\alpha_{\text{em}}^2$ **[1 pt]** $\sim \frac{9}{8} \times 0.12^2/137^{-2} \sim 300$ if we neglect the two s-channel diagrams, as instructed. (Generically, we'd expect those to contribute about the same amount to the ratio, per diagram, leading to $300 \to 300(1+2/6)^2) \sim 500$. **[Up to 1 bonus pt for estimate of ratio $\mathcal{O}(10^2)$, and some comment on expected size of neglected diagrams]**.

Problem 3

a) It is only the vector boson mass term that spoils the gauge invariance **[0.5 pt]**. We thus want to add the field ϕ as follows **[1 pt; only 0.5 if kinetic term of ϕ is not correctly normalized]**:

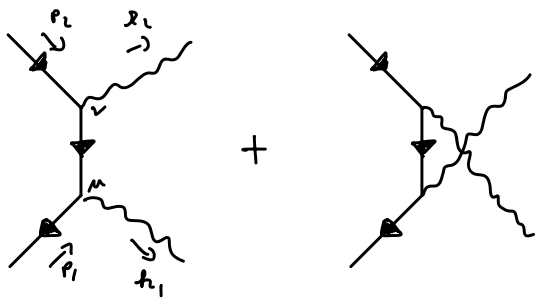
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_B^2\left(B_\mu + \frac{1}{m_B}\partial_\mu\phi\right)^2 + \overline{\psi}(i\not{D} - m_\psi)\psi \,, \tag{5}$$

This Lagrangian is invariant under

$$\psi \to (1+i\alpha)\psi \,, \textbf{[0.5 pt]} \tag{6}$$
$$B_\mu \to B_\mu + g^{-1}\partial_\mu\alpha \textbf{[0.5 pt]} \tag{7}$$
$$\phi \to \phi - (m_B/g)\,\alpha \,, \textbf{[0.5 pt]} \tag{8}$$

for arbitrary space-time functions $\alpha(x) \ll 1$. The Lagrangian stated above furthermore shows a *shift symmetry* $\phi \to \phi + const.$, which would be spoiled by introducing a mass term for ϕ. If ϕ arises as the Goldstone boson of a spontaneously broken symmetry, this is forbidden by Goldstone's theorem; if introduced by hand (referred to as Stueckelberg mechanism), there is however no fundamental reason why this symmetry should be there **[1 pt for any reasonable explanation mentioning shift symmetry or Goldstone's theorem]**.

b) The two diagrams that contribute to this process are **[1 pt]**:



The amplitude is hence given by **[1 pt]**

$$
\begin{aligned}
i\mathcal{M}_{\mu\nu}\epsilon^{\mu}(k_1)^*\epsilon^{\nu}(k_2)^* &= \bar{v}(p_1)\Bigg\{(-ig\gamma^{\mu})\frac{i(\not{p}_2-\not{k}_2)+m_\psi}{(p_2-k_2)^2-m_\psi^2}(-ig\gamma^{\nu})+ \\
&\qquad (-ig\gamma^{\nu})\frac{i(\not{p}_2-\not{k}_1+m_\psi)}{(p_2-k_1)^2-m_\psi^2}(-ig\gamma^{\mu})\Bigg\}u(p_2)\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2) \\
&= ig^2\bar{v}(p_1)\left\{\frac{\gamma^{\mu}(\not{p}_2-\not{k}_2+m_\psi)\gamma^{\nu}}{2\,p_2\cdot k_2-m_B^2}+\frac{\gamma^{\nu}(\not{p}_2-\not{k}_1+m_\psi)\gamma^{\mu}}{2\,p_2\cdot k_1-m_B^2}\right\}u(p_2)\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)\,.
\end{aligned}
\tag{9}
$$

We can now use the Dirac algebra to commute $\not{p}_2$ past the second gamma matrix, and then the Dirac equation $\not{p}u(p)=m_\psi u(p)$, to simplify this expression to

$$
\mathcal{M}_{\mu\nu} = g^2\bar{v}(p_1)\left\{\frac{-\gamma^{\mu}\not{k}_2\gamma^{\nu}+2\gamma^{\mu}p_2^{\nu}}{2\,p_2\cdot k_2-m_B^2}+\frac{-\gamma^{\nu}\not{k}_1\gamma^{\mu}+2\gamma^{\nu}p_2^{\mu}}{2\,p_2\cdot k_1-m_B^2}\right\}u(p_2)\,. \textbf{[1 pt]} \tag{10}
$$

We hence find

$$
\begin{aligned}
\mathcal{M}_{\mu\nu}k_1^{\mu} &= g^2\bar{v}(p_1)\left\{\frac{-\not{k}_1\not{k}_2\gamma^{\nu}+2\not{k}_1p_2^{\nu}}{2\,p_2\cdot k_2-m_B^2}+\frac{-\gamma^{\nu}\not{k}_1\not{k}_1+2\gamma^{\nu}(p_2\cdot k_1)}{2\,p_2\cdot k_1-m_B^2}\right\}u(p_2) \qquad (11)\\
&= g^2\bar{v}(p_1)\left\{\frac{-(\not{p}_1+\not{p}_2)\not{k}_2\gamma^{\nu}+2(\not{p}_1+\not{p}_2-\not{k}_2)p_2^{\nu}+m_B^2\gamma^{\nu}}{2\,p_2\cdot k_2-m_B^2}+\gamma^{\nu}\right\}u(p_2)\,,
\end{aligned}
$$

using momentum conservation and $\not{k}_i\not{k}_i=(k_i)^2=m_B^2$. We can now write $\not{p}_2\not{k}_2\gamma^{\nu}=2(p_2\cdot k_2)\gamma^{\nu}-\not{k}_2\not{p}_2\gamma^{\nu}=2(p_2\cdot k_2)\gamma^{\nu}-2\not{k}_2p_2^{\nu}+\not{k}_2\gamma^{\nu}\not{p}_2$. Using as before the Dirac equation to replace $\not{p}_2$, and $\not{p}_1v(p_1)=-m_\psi v(p_1)$ to replace p_1, we can thus use

$$
\begin{aligned}
& -(\not{p}_1+\not{p}_2)\not{k}_2\gamma^{\nu}+2(\not{p}_1+\not{p}_2-\not{k}_2)p_2^{\nu} && (12)\\
\rightarrow\ & m_\psi\not{k}_2\gamma^{\nu}-2(p_2\cdot k_2)\gamma^{\nu}+2\not{k}_2p_2^{\nu}-\not{k}_2\gamma^{\nu}m_\psi-2\not{k}_2p_2^{\nu}=-2(p_2\cdot k_2)\gamma^{\nu} && (13)
\end{aligned}
$$

in the second line of Eq. (11). This leaves us with $\mathcal{M}_{\mu\nu}k_1^{\mu}=g^2\bar{v}(p_1)\{-\gamma^{\nu}+\gamma\}u(p_2)=0$ **[1 more pt for arriving at final result]**. This is expected as a consequence of the *Ward identity*, according to which the amplitude must vanish when the polarization vector of an external photon is replaced with its momentum. Even for $m_B^2>0$, in particular, the gauge bosons still couple to the same conserved current $j^{\mu}=\bar{\psi}\gamma^{\mu}\psi$ as in QED, so the Ward identity (deriving from $\partial_\mu j^{\mu}=0$) is not affected. One conclusion is that the interaction term in this case still only couples to the transverse degrees of freedom of B^{μ}, even though the longitudinal polarization for a massive vector is also physical **[1 pt for any reasonable explanation]**.

c) The form of the covariant derivative indicates that U should transform as

$$
U\rightarrow\tilde{U}\equiv e^{\frac{i}{v}\tilde{\pi}^at^a}=e^{i\alpha^at^a}U \tag{14}
$$

for any space-time functions $\alpha^a(x)$, which in turn requires π^a to transform as

$$
\pi^a\rightarrow\tilde{\pi}^a=\pi^a+v\alpha^a \tag{15}
$$

By construction, $D_\mu U$ transforms in the same way as U, so $(D_\mu U)^\dagger (D_\mu U)$ is invariant under this transformation **[1 pt]**.
From now on, we define for ease of notation $B_\mu \equiv B_\mu^a t^a$ and $\pi \equiv \pi^a t^a$. Expanding the Stückelberg field then gives **[0.5 pt each, for correct expansion up to $\mathcal{O}(\pi)$]**

$$U = \mathbf{1} + i\frac{\pi}{v} - \frac{1}{2}\frac{\pi^2}{v^2} + ... \tag{16}$$

$$D_\mu U = i\frac{\partial_\mu \pi}{v} - \frac{\{\pi, \partial_\mu \pi\}}{2v^2} + ... - igB_\mu \left(1 + i\frac{\pi}{v} - \frac{1}{2}\frac{\pi^2}{v^2} + ...\right) \tag{17}$$

The gauge boson mass is the constant in front of a B^2 term in the Lagrangian, so we only need to keep the term $D_\mu U \supset -igB_\mu$ in the above expansion. We thus get

$$\mathcal{L}_{\text{mass}} \supset v^2 \text{Tr}\left[g^2 B^2\right] = v^2 g^2 B_\mu^a B^{b\mu} \text{Tr}\left[t^a t^b\right] = v^2 g^2 C(r) B_\mu^a B^{a\mu}\,, \tag{18}$$

from which we can identify the gauge boson mass as $m_B = vg\sqrt{2C(r)}$ **[1 pt]**. This construction to generate gauge boson masses works for arbitrary representations r. If the origin of the π fields are Goldstone bosons arising from the spontaneous symmetry breaking of a scalar field Φ, it is the symmetry breaking pattern and representation of Φ that determines r – e.g. $C(r) = \frac{1}{2}$ in the SM, since this is based on the fundamental rep of $SU(2)$ **[1 pt for any reasonable explanation]**.

d) We need to look again at the expansion stated in Eq. (17), keeping all terms that can give rise to two factors of π and one factor of B *after squaring*:

$$D_\mu U \supset i\frac{\partial_\mu \pi}{v} - \frac{\{\pi, \partial_\mu \pi\}}{2v^2} - igB_\mu \left(1 + i\frac{\pi}{v}\right) \text{ . } \textbf{[2 pts]} \tag{19}$$

Note here that $\pi(\partial\pi) \neq (\partial\pi)\pi$. We then find, again being careful with the fact that π and B_μ do not commute,

$$\begin{aligned}(D_\mu U)^\dagger (D^\mu U) &\supset \left(-i\frac{\partial_\mu \pi}{v} - \frac{\{\pi, \partial_\mu \pi\}}{2v^2} + igB_\mu + g\frac{\pi}{v}B_\mu\right)\left(i\frac{\partial^\mu \pi}{v} - \frac{\{\pi, \partial^\mu \pi\}}{2v^2} - igB^\mu + gB^\mu \frac{\pi}{v}\right) \\ &\supset i\frac{g}{v^2}\left(-(\partial_\mu \pi)B^\mu \pi + \frac{1}{2}\{\pi, \partial_\mu \pi\}B^\mu - \frac{1}{2}B_\mu \{\pi, \partial^\mu \pi\} + B^\mu (\partial_\mu \pi)\pi\right)\,.\end{aligned} \tag{20}$$

We can now use the cyclic property of the *trace* to move the B^μ left-most in all terms:

$$\begin{aligned}\text{Tr}\left[(D_\mu U)^\dagger (D^\mu U)\right] &\supset -i\frac{g}{v^2}\text{Tr}\left[B^\mu \pi(\partial_\mu \pi) - \frac{1}{2}B^\mu \{\pi, \partial_\mu \pi\} + \frac{1}{2}B_\mu \{\pi, \partial^\mu \pi\} - B^\mu (\partial_\mu \pi)\pi\right] \\ &= -i\frac{g}{v^2}\text{Tr}\left[B^\mu [\pi, \partial_\mu \pi]\right] \textbf{[1 pts]}\end{aligned} \tag{21}$$

The interaction term that was asked for is thus given by

$$\mathcal{L}_{\text{mass}} \supset -igB_\mu^a \pi^b (\partial^\mu \pi)\text{Tr}\left[t^a \left[t^b, t^c\right]\right] \tag{22}$$

$$= gB_\mu^a \pi^b (\partial^\mu \pi) f^{bcd} \text{Tr}\left[t^a t^d\right] \tag{23}$$

$$= gf^{abc} C(r) B_\mu^a \pi^b (\partial^\mu \pi^c)\,. \textbf{[1 pt]} \tag{24}$$

However, the fields π^a do not have canonically normalized kinetic terms. Looking at Eq. (17), we can most easily obtain those by replacing $B^a_\mu \to -(\partial_\mu \pi^a)/(gv)$ in Eq. (18). This gives

$$\mathcal{L}_{\text{mass}} \supset C(r)(\partial_\mu \pi^a)^2 \,. \tag{25}$$

In order to have canonically normalized kinetic terms, we hence must define $G^a \equiv \sqrt{2C(r)}\pi^a$ **[1 pt]** for the physical Goldstone modes. This leads to the interaction term

$$\mathcal{L}_{\text{mass}} \supset \frac{g}{2} f^{abc} B^a_\mu G^b (\partial^\mu G^c) \,, \tag{26}$$

which hence does *not* depend on the representation r **[1 pt]**.