Torsten Bringmann (torsten.bringmann@fys.uio.no)
http://www.uio.no/studier/emner/matnat/fys/FYS5120/v23/

Lecture spring 2023:

Quantum field theory II

Compulsory problem set – solutions

Problem I

a) This is just QED with two fermion fields, each of them needing their own counterterms for mass and field strength renormalization, respectively. At $\mathcal{O}(\alpha)$, in particular, there cannot be any diagrams with *both* ψ_a and ψ_b. As we have shown in the lecture, furthermore, gauge invariance implies that the charge renormalization must be the same for all fermions charged under the $U(1)$, related to the electromagnetic field strength renormalization by $Z_{e,a} = Z_{e,b} = 1/\sqrt{Z_3}$ (or $Z_{1,a}/Z_{2,a} = Z_{1,b}/Z_{2,b} = 1$). This implies $\delta_1 = \delta_2$, i.e. these two counterterms are not independent of each other. Eqs. (19.66) in Schwarz state the QED counterterms in the MS scheme; for $\overline{\rm MS}$, we just have to add $(\ln(4\pi) - \gamma_{\rm E})$ times the coefficient of $\ln\tilde{\mu}^2$ terms in the finite part – see Eqs. (19.25,19.29). In total:

$$(\delta_{1,a} \equiv)\delta_{2,a} = -\frac{1}{4}\frac{\alpha}{\pi}\left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma_{\rm E}\right] = \delta_{2,b}(\equiv \delta_{1,b}) \tag{1}$$

$$\delta_{m,a} = -\frac{3}{4}\frac{\alpha}{\pi}\left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma_{\rm E}\right] = \delta_{m,b} \tag{2}$$

$$\delta_3 = 2\cdot\left(-\frac{1}{3}\frac{\alpha}{\pi}\right)\left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma_{\rm E}\right] \tag{3}$$

Note that the equalities $\delta_{1,a} = \delta_{1,b}$ and $\delta_{m,a} = \delta_{m,b}$ do not necessarily hold at higher orders, and in particular depend on the renormalization scheme (while $\delta_1 = \delta_2$ is generally true). Hence, we have 5 truly independent counter terms. The additional factor of 2 for δ_3, arises from the fact that we have twice as many diagrams as in the QED case (both fermions could propagate in the loop).

b) To leading order, the relation between pole and $\overline{\rm MS}$ masses of ψ_a and ψ_b is just given by a fermion self-energy diagram with a single vector boson exchange – i.e. the same as in QED. We derived this relation in the lecture in the MS scheme, and for $\overline{\rm MS}$ we just have to replace $\tilde{\mu} \to \mu$:

$$\frac{m_{\overline{\rm MS},a}}{m_{p,a}} = 1 - \frac{\alpha}{\pi}\left(1 + \frac{3}{4}\log\frac{\mu^2}{m_{p,a}^2}\right), \tag{4}$$

and the same for $a \to b$. To leading order in α, we can invert this as

$$\frac{m_{p,a}}{m_{\overline{\rm MS},a}} = 1 + \frac{\alpha}{\pi}\left(1 + \frac{3}{4}\log\frac{\mu^2}{m_{\overline{\rm MS},a}^2}\right). \tag{5}$$

Note, however, that the cases 'a' and 'b' correspond to two completely independent processes, so we could use different renormalization conditions for the ψ_a and ψ_b self-energies, respectively. For the $\overline{\text{MS}}$ scheme, this implies that μ can be chosen differently – often, one would choose μ not too far away from the pole mass in order to avoid large logs at moderate energy scales (which can become quite relevant in particular for quark masses).

The interactions in this theory imply that at every vertex the number of ψ_a (anti-)fields changes by units of 2, or 0, and the same goes for ψ_b. A mass mixing would require a change of the number of both ψ_a and ψ_b by *one*, so this is not possible even for arbitrarily complicated diagrams.

c) The theory only contains dimension-less coupling constants (as well as mass terms); as we have seen in the lectures, this implies that it is renormalizable. A diagram with three external gauge bosons would at leading order be generated by a fermion loop (either only ψ_a or only ψ_b). The superficial degree of divergence of such a diagram is $D = 1$, so if the diagram did not vanish the expected divergent terms ($\propto \log\Lambda$ and $\propto \Lambda$) would have to be cancelled by a counter term in $\mathcal{L}_\psi$ containing three A. Such a term, however, cannot be part of the Lagrangian as it would violate gauge invariance.
For the 5-point function, we have five fermion propagators and hence $D = -1$ – so no counterterms are needed, and the above argument does not work. One can instead invoke *Furry's theorem*, which in fact forbids *any* correlation function in QED of an odd number of photons (and no fermions).[1]

Problem II

a) Both ϕ^2 and ϕ^4 operators are generated at 1-loop level (via fermion loops, with an equal number of ψ_a and ψ_b propagators). These diagrams have superficial degrees of divergence of $D = 2$ and $D = 0$, respectively, and there is in fact no symmetry that would force scalar 2- or 4-point amplitudes to be zero (see also IIb and IIc below). Those loops thus diverge, and are not calculable – they are genuine parameters of the theory and must be determined by a measurement. So we have to add both a mass and a self-interaction term to the Lagrangian,

$$\mathcal{L} \to \mathcal{L} - \frac{1}{2} m_\phi^2 \phi^2 - \frac{\lambda}{4!}\phi^4 \,. \tag{6}$$

Renormalizing the bare Lagrangian parameters as $m_\phi^{(0)} \equiv \sqrt{Z_{m,\phi}}\, m_\phi$ and $\lambda^{(0)} \equiv Z_\lambda \lambda$, we further have to add the following counterterms:

$$\mathcal{L}^{\text{c.t.}} \to \mathcal{L}^{\text{c.t.}} - \frac{1}{2}(\delta_{m,\phi} + \delta_\phi)\phi^2 - \frac{1}{4!}(\delta_\lambda + 2\delta_\phi)\phi^4 \,. \tag{7}$$

[1] In a nutshell, the proof proceeds by noting that the Lagrangian is invariant under C, while $A^\mu \to -A^\mu$ under this discrete symmetry. Hence, any amplitude with an odd number of photons picks up a minus sign under C (also $\overline{\psi}\gamma^\mu\psi$ does, thus allowing the standard covariant derivative).

Here, we have as usual defined $\delta_\phi \equiv Z_\phi - 1$, and

$$\begin{aligned} \delta_{m,\phi} &\equiv Z_\phi Z_{m,\phi} - 1 - \delta_\phi = Z_{m,\phi} - 1 + \mathcal{O}(\lambda^4) && (8) \\ \delta_\lambda &\equiv Z_\phi^2 Z_\lambda - 1 - 2\delta_\phi = Z_\lambda - 1 + \mathcal{O}(\lambda^4)\,. && (9) \end{aligned}$$

b) On top of the 5 counter terms from problem I, we now additionally have the three counterterms mentioned in IIa, as well as the one for the Yukawa coupling,

$$\delta_y \equiv \sqrt{Z_{2,a}}\sqrt{Z_{2,b}} - 1 - \frac{1}{2}\delta_{2,a} - \frac{1}{2}\delta_{2,b} = Z_y - 1 + \mathcal{O}(\lambda^4, g^2\lambda^2)\,. \qquad (10)$$

Out of these, only $\underline{\delta_3}$ is the same form as in problem I, because at 1-loop level it is still only the same fermion loop that contributes to $\langle AA\rangle$. (Also, $\delta_1 = \delta_2$ has to remain true because of gauge invariance.)

Next, we calculate $\underline{\delta_2 \text{ and } \delta_m}$, noting that now there is an additional contribution to the fermion self-energy – from a diagram with a scalar rather than a vector loop. We obtain this directly from (18.8) by changing vector to scalar couplings:

$$\begin{aligned} i\Sigma^\phi_{2,a}(\not{p}) &= (iy)^2 \int \frac{d^4k}{(2\pi)^2} \frac{i(\not{k} + m_b)}{k^2 - m_b^2 + i\epsilon} \frac{i}{(k-p)^2 - m_\phi^2 + i\epsilon} && (11) \\ &= y^2 \int \frac{d^4k}{(2\pi)^2} \int_0^1 dx\, \frac{\not{k} + m_b}{(k^2 - m_b^2)(1-x) + (k-p)^2 x - m_\phi^2 x + i\epsilon} && (12) \\ &= y^2 \int \frac{d^4k}{(2\pi)^2} \int_0^1 dx\, \frac{\not{p}x + m_b}{\left[k^2 - \Delta + i\epsilon\right]^2} && (13) \\ &= i\frac{y^2}{4\pi^2}\left(\frac{1}{4\epsilon}\not{p} + \frac{1}{2\epsilon} m_b\right) + \text{finite}\,, && (14) \end{aligned}$$

where $\Delta = -p^2 x(1-x) + m_b^2(1-x) + m_\phi^2 x$, after completing the square with $k \to k + px$, and the corresponding result holds for $a \leftrightarrow b$. The last step follows by comparison with (18.13). With

$$\Sigma_{2,a} = \delta_{2,a}\not{p} - (\delta_{2,a} + \delta_{m,a}) m_a + \Sigma^\gamma_{2,a}(\not{p}) + \Sigma^\phi_{2,a}(\not{p})\,, \qquad (15)$$

and the divergent part of $\Sigma^\gamma_{2,a}(\not{p})$ cancelled by Eq. (1), we can first look at the terms $\propto \not{p}$ to read of

$$\delta_{2,a} = \left(-\frac{1}{4}\frac{e^2}{4\pi^2} - \frac{1}{8}\frac{y^2}{4\pi^2}\right)\left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma_\text{E}\right] = \delta_{2,b}\,. \qquad (16)$$

With this, making the constant parts in Eq. (15) finite requires

$$\delta_{m,a/b} = \left\{-\frac{3}{4}\frac{e^2}{4\pi^2} + \frac{1}{8}\left(1 + 2\frac{m_{b/a}}{m_{a/b}}\right)\frac{y^2}{4\pi^2}\right\}\left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma_\text{E}\right]\,. \qquad (17)$$

We next take a look at the scalar self-energy, from which we can calculate δ_ϕ and $\delta_{m,\phi}$. From Eq. (7), it takes the form

$$i\Pi_\phi(p^2) = i\delta_\phi p^2 - i(\delta_{m,\phi} + \delta_\phi)m_\phi^2 + i\Pi_\phi^{\rm loop}(p^2)\,, \tag{18}$$

where (at lowest order) both a fermion and a scalar loop contribute to

$$i\Pi_\phi^{\rm loop}(p^2) = i\Pi_\phi^{\rm fermion}(p^2) + i\Pi_\phi^{\rm scalar}(p^2)\,. \tag{19}$$

Unlike in the case of QED, there are *two* fermion loops, which however are numerically the same. Taking into account this factor of 2, we find (similar to the photon self-energy)

$$\begin{aligned} i\Pi_\phi^{\rm fermion}(p^2) &= 2(iy)^2(-1){\rm Tr}\left[\int \frac{d^4k}{(2\pi)^4}\frac{i(\not k + m_b)}{k^2-m_b^2+i\epsilon}\frac{i(\not k - \not p + m_a)}{(k-p)^2-m_a^2+i\epsilon}\right] \qquad (20)\\ &= -2y^2 d\int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\frac{k^2 - k\cdot p + m_a m_b}{\left[(k^2-m_b^2)(1-x)+((k-p)^2-m_a^2)x + i\epsilon\right]^2}\\ &\overset{k\to k+px}{=} -2y^2 d\int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\frac{k^2 - p^2x(1-x) + m_a m_b}{\left[(k^2-\Delta+i\epsilon\right]^2}\,, \qquad (21)\end{aligned}$$

where $\Delta = -p^2x(1-x) + m_b^2(1-x) + m_a^2 x$. For $\overline{\rm MS}$, we only need the divergent parts. Using (B.36,B.37), we can thus replace $\Gamma(2-d/2) \to 2/\epsilon$ and $\Gamma(1-d/2) \to -2/\epsilon$, and otherwise use $d \to 4$. Thus, we find

$$\begin{aligned} i\Pi_\phi^{\rm fermion}(p^2) &= -y^2 8\frac{2}{\epsilon}\int_0^1 dx\left[2\frac{i}{(4\pi)^2}\Delta - \left(p^2x(1-x) - m_a m_b\right)\frac{i}{(4\pi)^2}\right] + \text{finite}\\ &= i\frac{y^2}{2\pi^2}\frac{2}{\epsilon}\int_0^1 dx\left[2(x-1)m_b^2 - 2xm_a^2 - m_a m_b - 3p^2x(1-x)\right] + \text{finite}\\ &= i\frac{y^2}{2\pi^2}\frac{2}{\epsilon}\left[-(m_a^2 + m_a m_b + m_b^2) + \frac{p^2}{2}\right] + \text{finite}\,. \qquad (22)\end{aligned}$$

The scalar loop is much simpler. Using (B.35) with $a = 0$, $b = 1$ and $\Delta = m_\phi^2$, we find (note the symmetry factor 1/2 !)

$$\begin{aligned} i\Pi_\phi^{\rm scalar}(p^2) &= -i\lambda\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_\phi^2+i\epsilon} \qquad (23)\\ &= -\frac{\lambda}{2}\frac{i}{(4\pi)^2}m_\phi^2\frac{\Gamma(2)\Gamma(1-d/2)}{\Gamma(1)\Gamma(2)} = i\frac{m_\phi^2}{8}\frac{\lambda}{4\pi^2}\frac{2}{\epsilon}\,. \qquad (24)\end{aligned}$$

Comparing Eqs. (22,24) to Eq. (18), we can thus identify

$$\begin{aligned} \delta_\phi &= -\frac{y^2}{4\pi^2}\left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma_{\rm E}\right] \qquad (25)\\ \delta_{m,\phi} &= \left[\frac{1}{8}\frac{\lambda}{4\pi^2} - \frac{y^2}{4\pi^2}\left(2\frac{m_a^2 + m_a m_b + m_b^2}{m_\phi^2} - 1\right)\right]\times\left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma_{\rm E}\right] \qquad (26)\end{aligned}$$

c) The theory has a (slightly hidden) discrete 'Z_2' symmetry:

$$\phi \to -\phi, \qquad \psi_{a,b} \to e^{-i\beta_{a,b}}\psi_{a,b} \quad \text{with } e^{i(\beta_a-\beta_b)} = -1\,. \tag{27}$$

This means that amplitudes can have an arbitrary number of external fermion *pairs* $\psi_a\psi_a$ (or $\psi_b\psi_b$), independently of the number of external scalars (or vectors); an additional unpaired ψ_a must come together with an unpaired ψ_b *and* an odd number of scalars ϕ.

The following dim ≤ 4 terms are Lorentz-invariant, but cannot be generated:

- dim 1: ϕ would violate (27).
- dim 2: $\partial_\mu A^\mu$ & $A_\mu A^\mu$ would violate gauge invariance.
- dim 3: $\overline{\psi}_a\psi_b$ & ϕ^3 would violate (27).
- dim 4: $\partial_\mu A_\nu F^{\mu\nu}$, A^4 etc. would violate gauge invariance.
- dim 4: $\partial_\mu\phi A^2$, $\phi^2 A^2$ etc. would also violate gauge invariance.[2]
- dim 4: $\phi\overline{\psi}_a\psi_a$ & $\phi\overline{\psi}_b\psi_b$ would violate (27).

The theory only has dim ≤ 4 operators and is hence renormalizable. If a dim > 4 operator existed as a Lagrangian parameter, the theory would not be renormalizable. Hence, all dim > 4 operators must be calculable (i.e. functions of the Lagrangian parameters) and finite.

Problem III

a) For $m_a = m_b = 0$, the theory exhibits an enhanced (chiral) symmetry that acts as a custodial symmetry:

$$P_L\psi_{a,b} \;\to\; e^{i\beta_L}P_L\psi_{a,b}\,, \tag{28}$$

$$P_R\psi_{a,b} \;\to\; e^{i\beta_R}P_R\psi_{a,b}\,. \tag{29}$$

Note that, as a consequence of the Yukawa interaction, a and b have to transform in the same way under these transformations. We could in fact have expected a custodial symmetry from the result for $\delta_{m,a}$ and $\delta_{m,b}$ – showing that radiative corrections to the fermion masses are proportional to $m_{a,b}$ themselves.

b) For a single fermion loop diagram, with vanishing fermion masses, we have (where p_1 & p_2 are ingoing, p_3 & p_4 outgoing momenta)

$$\begin{aligned} i\mathcal{M} &= -(iy)^4\mathrm{Tr}\left[\int \frac{d^4k}{(2\pi)^4}\frac{ik\!\!\!/}{k^2+i\epsilon}\frac{i(k\!\!\!/+p\!\!\!/_1)}{(k+p_1)^2+i\epsilon}\frac{i(k\!\!\!/+p\!\!\!/_1-p\!\!\!/_3)}{(k+p_1-p_3)^2+i\epsilon}\frac{i(k\!\!\!/-p\!\!\!/_2)}{(k-p_2)^2+i\epsilon}\right] \\ &\overset{k\to\infty}{\sim} -y^4\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\,[k\!\!\!/k\!\!\!/k\!\!\!/k\!\!\!/]}{k^8+i\epsilon} \qquad (30) \\ &= -y^4 d\;\mu^\epsilon\int\frac{d^dk}{(2\pi)^4}\frac{1}{k^4+i\epsilon} = -i\frac{y^4}{4\pi^2}\frac{2}{\epsilon} \qquad (31) \end{aligned}$$

[2]Note that a term like $|D\phi|^2$ *would* be allowed by gauge invariance *if* ϕ was complex. A real scalar field (or in fact any field that is its own conjugate) cannot carry any charge.

For the entire amplitude, we have to multiply by a factor 4 from $a \leftrightarrow b$ and reversing the fermion arrow direction (this holds also without taking the limit $k \to \infty$). Further independent(!) diagrams arise from exchanging $p_1 \leftrightarrow p_2$ as well as $p_1 \leftrightarrow p_3$. In total there are thus 12 different diagrams – which however all contribute the same amount to the divergent part of the amplitude.

For a single scalar loop diagram, we have (note the symmetry factor!)

$$i\mathcal{M} = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_\phi^2 + i\epsilon} \frac{i}{(k+p_1+p_2)^2 - m_\phi^2 + i\epsilon} \quad (32)$$

$$\overset{k\to\infty}{\sim} \frac{\lambda^2}{2}\mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^4 + i\epsilon} = i\frac{1}{8}\frac{\lambda^2}{4\pi^2}\frac{2}{\epsilon} \quad (33)$$

This result has to be multiplied by 3, since there are three different diagrams that we can get by exchanging $p_2 \to p_3$ and $p_2 \to p_4$, respectively.

c) From b) and Eq. (7) we find for the counterterm

$$\delta_\lambda + 2\delta_\phi = \left(12\frac{y^4}{4\pi^2} - \frac{3}{8}\frac{\lambda^2}{4\pi^2}\right) \times \left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma_{\rm E}\right]. \quad (34)$$

The μ dependence (in $\overline{\rm MS}$) comes entirely from the $\mu^\epsilon = 1 + \frac{\epsilon}{2}\log\mu^2 + \mathcal{O}(\epsilon^2)$ term in front of $d-$dimensional integrals. In other words, when adding the matrix elements from b) to the counter term above, and also including the tree-level result, we find

$$i\mathcal{M} \equiv -i\Lambda = -i\lambda + i\left(12\frac{y^4}{4\pi^2} - \frac{3}{8}\frac{\lambda^2}{4\pi^2}\right)\ln\frac{\mu^2}{s_0} + X\,, \quad (35)$$

where X and s_0 are finite functions of external momenta, independent of μ.

d) The renormalization condition fixes $\Lambda = \lambda$ (for a given y) at some scale s_0, i.e.

$$X(s_0) = -i\left(12\frac{y^4}{4\pi^2} - \frac{3}{8}\frac{\lambda^2}{4\pi^2}\right)\ln\frac{\mu^2}{s_0}\,. \quad (36)$$

The predicted four-point coupling as measured at another scale s_0' is instead

$$i\mathcal{M} \equiv -i\Lambda' = -i\lambda + i\left(12\frac{y^4}{4\pi^2} - \frac{3}{8}\frac{\lambda^2}{4\pi^2}\right)\ln\frac{\mu^2}{s_0'} + X(s_0')\,, \quad (37)$$

Using Eq. (36), we thus get an expression independent of μ:

$$\Lambda' \simeq \lambda + \left(12\frac{y^4}{4\pi^2} - \frac{3}{8}\frac{\lambda^2}{4\pi^2}\right)\ln\frac{s_0'}{s_0} + iX(s_0') - iX(s_0)\,, \quad (38)$$

Note that X and X' are also second-order in perturbation theory. Being dimensionless functions of external momenta, their difference should tend to 0 as $|s_0' - s_0| \to \infty$.

e) For $p_i^\mu = (m_\phi, \mathbf{0})$, i.e. identical external momenta, we can actually fully compute the loop integrals, because all diagrams are identical. Let us start with the fermion loop. Here we first note that we can simplify the trace as

$$\begin{aligned} \mathrm{Tr}\left[\not{k}(\not{k}+\not{p}_1)\not{k}(\not{k}-\not{p}_1)\right] &= \mathrm{Tr}\left[\not{k}\not{k}(\not{k}+\not{p}_1)(\not{k}-\not{p}_1)\right] - 2k\cdot p_1 \mathrm{Tr}\left[\not{k}(\not{k}-\not{p}_1)\right] \\ &= d\,k^2(k^2-p_1^2) - 2d\,k\cdot p_1(k^2 - k\cdot p_1) \end{aligned} \tag{39}$$

Hence, we have (note the way of re-writing the denominators before attempting Feynman parametrization !)

$$\begin{aligned} i\mathcal{M} &= -12d\; y^4\mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \left\{ \frac{1}{k^2(k^2-m_\phi^2)+i\epsilon} - \underbrace{\frac{2k\cdot p_1(k^2-k\cdot p_1)}{k^4(k^2-m_\phi^2)^2+i\epsilon}}_{\to -2p_1^2 k^{-2}(k^2-m_\phi^2)^{-2}} \right\} \qquad (40) \\ &\equiv i\mathcal{M}_a + i\mathcal{M}_b\,, \qquad (41) \end{aligned}$$

where

$$\begin{aligned} i\mathcal{M}_a &= -12d\; y^4\mu^\epsilon \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[(1-x)k^2 + x(k^2-m_\phi^2)+i\epsilon\right]^2} \qquad (42) \\ &= -12d\; y^4\mu^\epsilon \int_0^1 dx \underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 - xm_\phi^2 + i\epsilon\right]^2}}_{i(4\pi)^{-d/2}(xm_\phi^2)^{-\epsilon/2}\Gamma(\epsilon/2)} \qquad (43) \\ &= -3i\frac{y^4}{4\pi^2}\left(\frac{2}{\epsilon} - \gamma_E\right)(4-\epsilon) \underbrace{\int_0^1 dx \left(\frac{4\pi\mu^2}{xm_\phi^2}\right)^{\epsilon/2}}_{1+\frac{\epsilon}{2}\left[1+\ln(4\pi\mu^2/m_\phi^2)\right] \quad \text{[Wolfram Alpha :)]}} \qquad (44) \\ &= -12i\frac{y^4}{4\pi^2}\left\{\frac{2}{\epsilon} + \frac{1}{2} + \ln\left(\frac{4\pi\mu^2}{m_\phi^2}\right) - \gamma_E\right\} \qquad (45) \end{aligned}$$

and a finite contribution

$$\begin{aligned} i\mathcal{M}_b &= 24d\; y^4 m_\phi^2 \mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2(k^2-m_\phi^2)^2+i\epsilon} \qquad (46) \\ &= 96\; y^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k^2-1)^2+i\epsilon} \qquad (47) \\ &= 96\; y^4 \int_0^1 dx \int_0^1 dy \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{2!}{\left[k^2 - x - y + i\epsilon\right]^3}}_{-i(4\pi)^{-2}/(x+y)} \qquad (48) \\ &= -24i\frac{y^4}{4\pi^2}\int_0^1 dx \int_0^1 dy \frac{1}{x+y} = -48i\frac{y^4}{4\pi^2}\ln 2 \qquad (49) \end{aligned}$$

For the scalar loop, we get

$$
\begin{aligned}
i\mathcal{M} &= \frac{3}{2}\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_\phi^2 + i\epsilon} \frac{1}{(k+2p_1)^2 - m_\phi^2 + i\epsilon} && (50)\\
&= \frac{3}{2}\lambda^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[k^2 + 4x(k\cdot p_1 + p_1^2) - m_\phi^2 + i\epsilon\right]^2} && (51)\\
&= \frac{3}{2}\lambda^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[k^2 - m_\phi^2(1-2x)^2 + i\epsilon\right]^2} && (52)\\
&= i\frac{3}{8}\frac{\lambda^2}{4\pi^2}\left(\frac{2}{\epsilon} - \gamma_E\right) \underbrace{\int_0^1 dx \left(\frac{4\pi\mu^2}{(1-2x)^2 m_\phi^2}\right)^{\epsilon/2}}_{1+\frac{\epsilon}{2}\left[1+\ln(4\pi\mu^2/m_\phi^2)\right]} && (53)\\
&= i\frac{3}{8}\frac{\lambda^2}{4\pi^2}\left\{\frac{2}{\epsilon} + 1 + \ln\left(\frac{4\pi\mu^2}{m_\phi^2}\right) - \gamma_E\right\} && (54)
\end{aligned}
$$

Adding to these amplitudes $i\times$ the counterterms *that we would get in* $\overline{\text{MS}}$, i.e. Eq. (34), we find most terms as expected to cancel. In total, the 4-point amplitude becomes

$$
i\Lambda \stackrel{!}{\equiv} i\lambda = i\lambda - 12i\frac{y^4}{4\pi^2}\left\{-\frac{1}{2} + 4\ln 2 + 1 + \ln\left(\frac{\mu^2}{m_\phi^2}\right)\right\} + i\frac{3}{8}\frac{\lambda^2}{4\pi^2}\left\{1 + \ln\left(\frac{\mu^2}{m_\phi^2}\right)\right\}, \tag{55}
$$

where the first definition (!) implements our renormalization condition. In other words, this renormalization condition can indeed be realized in the $\overline{\text{MS}}$ scheme, namely by choosing μ such that

$$
0 = (6 - 4\ln 2)y^4 + \left(\frac{3}{8}\lambda^2 - 12y^4\right)\left\{1 + \ln\left(\frac{\mu^2}{m_\phi^2}\right)\right\}, \tag{56}
$$

or

$$
1 + \ln\left(\frac{\mu^2}{m_\phi^2}\right) = \frac{\frac{3}{8}(6 - 4\ln 2)y^4}{32y^4 - \lambda^2} = \mathcal{O}(1)\,. \tag{57}
$$

In other words, one (typically) needs $\mu^2 \sim \mathcal{O}(s = 4m_\phi^2)$. Relations like this motivate statements such as that one 'should', in dimensional renormalization, choose μ to correspond to a relevant energy scale of the process.