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FYS5120: Path integral quantization

In QFT1, you have learned the *canonical quantization* procedure for classical fields. Advantages of this procedure include the fact that it is (in principle) straight-forward to perform, tightly connected to what you know from ordinary quantum mechanics, and that it leads to a relatively intuitive interpretation – namely the entire Hilbert space being spanned by multi-'particle' states (where 'particles' are simply field quanta with definite energies and momenta, satisfying the relativistic dispersion relation).

Unfortunately, the procedure of deriving Feynman rules – which are needed for efficient rate calculations – is rather cumbersome in this approach. On top of that, the canonical quantization approach fails for only slightly more complicated theories than what you have encountered so far; as you have seen, in fact, already quantizing ordinary photons in QED poses a significant challenge. In this course we will therefore mostly use an alternative approach know as **path integral quantization**. From a practical point of view, this approach simply boils down to using an alternative **master formula** for computing n-point correlation functions:

$$\langle\Omega|T\left\{\phi(x_1)...\phi(x_n)\right\}|\Omega\rangle = \frac{\int \mathcal{D}\phi\,\phi(x_1)...\phi(x_n)\exp\left(iS[\phi]\right)}{\int \mathcal{D}\phi\,\exp\left(iS[\phi]\right)}\,. \tag{1}$$

As emphasized in the background notes, essentially all observables relevant to particle interactions – such as cross sections and decay rates, in particular – can be expressed in terms of these correlation functions. Any *algorithm* to compute the right-hand-side of the above expression can in this sense be taken as a *definition* of a QFT.

Here we will mostly follow this algorithmic approach to QFT. In some contrast to what you did in the context of canonical quantization, we will thus spend significantly more time on understanding how to *use* the above expression in practice than on 'deriving' (or rather: motivating) it. We will also happily skip over the fact that it is in fact rather challenging to define the *individual* ingredients in the Eq. (1) in a mathematically *fully* satisfactory way. This choice is not only due to practical reasons (lack of time in this course) but also a reflection of the fact that path integral quantization in this sense has become the 'gold standard' in the field: it reproduces exactly all correlation functions that can be derived in canonical quantization, allows to compute them in more complicated situations – and produces in all cases results in perfect agreement with experiment. For an empirical science like physics, it is obviously the latter aspect that is of ultimate importance.

1 Basic Interpretation

Before attempting any interpretation, let us first understand what is *meant* by the symbol '$\mathcal{D}$' on the r.h.s. of Eq. (1). For that, consider a spacetime function $\phi(x)$ and discretize it over a set of N spacetime points $\{x_i\}$, resulting in a set of pairs $\big\{(x_i,\phi_i)\big\}$

with $i \in [1, N]$ and $\phi_i \equiv \phi(x_i)$. The path integral over ϕ is then formally defined as the limit

$$\mathcal{D}\phi \equiv \lim_{N\to\infty} \mathcal{D}_N\phi \tag{2}$$

over the N-dimensional integral

$$\mathcal{D}_N\phi \equiv \prod_{i=1}^{N} d\phi_i \,. \tag{3}$$

In other words, the path integral is basically the *sum over all possible functions* $\phi(x)$ (while $\int \mathcal{D}_N$ would be the sum over all functions *discretized* over the net $\{x_i\}$), typically subject to some boundary conditions. As this definition admittedly looks rather daunting, two mitigating remarks are in order: *i)* one way to understand the introductory comments about the 'algorithmic approach' is to say that in implementing Eq. (1) we only ever use $\mathcal{D}_N$ in actual calculations, and then take the continuum limit of the result – so in some sense we never *actually* evaluate infinite-dimensional integrals. *ii)* We will only have to evaluate a very small number of path integrals *explicitly*; as we will see, all necessary results can then be expressed in a very simple way in terms of these 'basic' integrals.

Apart from the appearance of the path integral as a *technical* complication, there is a major *conceptual difference* between Eq. (1) and the corresponding master formula in canonical quantization: the fields ϕ here are in fact **classical fields**, i.e. ordinary space-time functions rather than field operators. This also explains why no expectation values appear on the r.h.s.. Likewise, S is the **classical action**. The path integral in the nominator thus sums over all[1] classical field configurations, exponentially weighted by the action for these configurations. From the method of stationary phases we know that integrals of the form appearing in Eq. (1) are (by far) dominated by contributions with $\delta S = 0$. On the other hand, $\delta S[\phi_{\rm cl}] = 0$ means that $\phi_{\rm cl}$ obeys the classical Euler-Lagrange equations of motion. In conclusion, path integrals are generally dominated by fields following classical trajectories – but also receive small further 'quantum' corrections.

1.1 Motivation in quantum mechanics

In quantum mechanics, the amplitude for an initial state $|i\rangle$ (describing a particle at position $\mathbf{x}_i$ at time t_i) to evolve into a final state $|i\rangle$ (describing a particle at position $\mathbf{x}_f$ at time t_f) is given by

$$\langle f|i\rangle = \langle \mathbf{x}_f|U(t_f, t_i)|\mathbf{x}_i\rangle \,, \tag{4}$$

where $U(t_f, t_i)$ is the time-evolution operator (which is difficult to bring into a simple form if the Hamiltonian has an explicit time-dependence). It turns out that this can

[1]Notably, this includes non-perturbative configurations, which are by construction not captured in the canonical quantization approach. This is another bonus of the path integral formulation, which however we have no time to further explore in this course.

also be written as

$$\langle f|i\rangle \;\propto\; \int_{\mathbf{x}(t_i)=\mathbf{x}_i}^{\mathbf{x}(t_f)=\mathbf{x}_f} \mathcal{D}\mathbf{x}(t)\exp\left(\frac{i}{\hbar}S[\mathbf{x},\dot{\mathbf{x}}]\right) \tag{5}$$

$$\equiv \lim_{N\to\infty}\int\left(\prod_{j=1}^{N}d\mathbf{x}_j\right)\prod_{k=1}^{N}\exp\left(\frac{i}{\hbar}L[\mathbf{x}_k,\dot{\mathbf{x}}_k]\delta t\right)\,, \tag{6}$$

where $\delta t \equiv (t_f - t_i)/N$ and $\dot{\mathbf{x}}_k = (\mathbf{x}_{k+1} - \mathbf{x}_k)/\delta t$. In other words, the positions $\mathbf{x}_k$ should be understood as corresponding to subsequent times $t_k = t_i + k\delta t$.

The above expression asserts that the transition amplitude can be interpreted by the **sum over** *all* **paths** from $\mathbf{x}_i$ to $\mathbf{x}_f$ – weighted by an expression that is by far dominated for the *specific* path $\mathbf{x}(t)$ for which $\delta S = 0$, i.e. the one expected by the classical Euler-Lagrange equations.[2] Expressing $\langle f|i\rangle$ in terms of a path integral, as above, thus generalizes the lesson learned from the famous double-slit experiment – where the most important outcome is that one has to sum the *amplitudes* (not probabilities) for the two possible paths. Here, instead of a single plate with a double slit, the expression implements very many (N) imaginary plates that are reached at time t_k, respectively, with infinitely many 'slits' each (represented by the fact that we integrate over all possible values of *each* of the positions $\mathbf{x}_k$, for all time steps).

'Proof'. Let us in the following explicitly indicate every *operator* with a hat, in order to mark the transition to classical quantities as clearly as possible. For infinitely small δt, as well as $\partial_t H = 0$, we have $U(t+\delta t, t) = \exp[-i\hat{H}(t)\delta t]$. Inserting many times the identity $\int d\mathbf{x}\,|\mathbf{x}\rangle\langle\mathbf{x}| = \mathbb{1}$, this gives

$$\begin{aligned} U(t_f,t_i) &= \int d\mathbf{x}_1...d\mathbf{x}_N\, e^{-i\hat{H}(t_N)\delta t}|\mathbf{x}_N\rangle\langle\mathbf{x}_N|e^{-i\hat{H}(t_{N-1})\delta t}|\mathbf{x}_{N-1}\rangle\langle\mathbf{x}_{N-1}|...e^{-i\hat{H}(t_1)\delta t}|\mathbf{x}_1\rangle\langle\mathbf{x}_1|e^{-i\hat{H}(t_i)\delta t} \\ &= \int d\mathbf{x}_1...d\mathbf{x}_N\, e^{-i\hat{H}(t_N)\delta t}|\mathbf{x}_N\rangle\langle N|N-1\rangle\langle N-1|N-2\rangle...\langle 2|1\rangle\langle\mathbf{x}_1|e^{-i\hat{H}(t_i)\delta t}\,, \end{aligned} \tag{7}$$

where

$$\langle k+1|k\rangle \equiv \langle\mathbf{x}_{k+1}|e^{-i\hat{H}(t_k)\delta t}|\mathbf{x}_k\rangle\,. \tag{8}$$

In other words, we decompose the time-evolution into many short steps, with the corresponding time-evolution operators evaluated at $t_k = t_i + k\delta t$ (with $t_N = t_f - \delta t$). The advantage of this decomposition is that, in contrast to the full time evolution, the transition amplitude for such *infinitesimal* time evolutions at time t_k is straight-forward to evaluate. In order to see this, we insert an identity in momentum space, $\int \frac{d\mathbf{p}}{(2\pi)^3}\,|\mathbf{p}\rangle\langle\mathbf{p}| = \mathbb{1}$, and use the Hamiltonian for a single particle in a time-dependent potential, $\hat{H}(t) = \hat{\mathbf{p}}^2/(2m) + V(\hat{\mathbf{x}},t)$,

$$\langle k+1|k\rangle = \int\frac{d\mathbf{p}}{(2\pi)^3}\underbrace{\langle\mathbf{x}_{k+1}|\mathbf{p}\rangle}_{e^{i\mathbf{p}\cdot\mathbf{x}_{k+1}}}\langle\mathbf{p}|e^{-i\frac{\hat{\mathbf{p}}^2}{2m}\delta t - iV(\hat{\mathbf{x}}_k,t_k)\delta t}|\mathbf{x}_k\rangle \tag{9}$$

$$= e^{-iV(\mathbf{x}_k,t_k)\delta t}\int\frac{d\mathbf{p}}{(2\pi)^3}\,e^{-i\frac{\mathbf{p}^2}{2m}\delta t - i\mathbf{p}(\mathbf{x}_{k+1}-\mathbf{x}_k)}\,. \tag{10}$$

Note that in the final step we not only used the standard $\hat{\mathbf{x}}|\mathbf{x}_k\rangle = \mathbf{x}_k|\mathbf{x}_k\rangle$ and $\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$, but also – and this is crucial – the fact that $\delta t/t \ll 1$. Namely, the latter implies that we can expand $\exp(-i\hat{H}) \approx \mathbb{1} - i\hat{H}$, which is an expression without ordering ambiguities (while in general, in contrast, those have to be taken properly into account based on $[\hat{\mathbf{x}},\hat{\mathbf{p}}] \neq 0$). The final integration can be used by performing standard Gaussian integrals of the form[3]

$$\int_{-\infty}^{\infty}dp\,e^{-\frac{a}{2}p^2+Jp} = \sqrt{\frac{2\pi}{a}}e^{J^2/2a}\,. \tag{11}$$

[2]Referring again to the method of stationary phases, the next-to-leading order corrections are suppressed with a factor of $\hbar$ with respect to the classical expectation.

[3]You may notice that this equation holds for real positive values of a, while we use it for complex values ($a = i\delta t/m$). The argument still works, by analytic continuation of δt into the complex domain ('Wick rotation'), effectively implementing $\delta t \to \delta t(1 - i\varepsilon)$ to make the integral converge.

Neglecting the pre-factor, we thus find

$$\langle k+1|k\rangle \propto e^{-iV(\mathbf{x}_k,t_k)\delta t} e^{-\frac{m}{2i\delta t}(\mathbf{x}_{k+1}-\mathbf{x}_k)^2} = e^{i\{\frac{m}{2}\dot{\mathbf{x}}_k^2 - V(\mathbf{x}_k,t_k)\}\delta t} = e^{iL(\mathbf{x}_k,\dot{\mathbf{x}}_k)\delta t}\,. \tag{12}$$

Somewhat miraculously, the Gaussian integral in Eq. (11) thus effectively implements an (inverse) **Legendre transformation** $H(\mathbf{x},\mathbf{p}) \to L(\mathbf{x},\dot{\mathbf{x}})$. Using this result, Eq. (7) becomes

$$\langle f|i\rangle \propto \int d\mathbf{x}_1 ... d\mathbf{x}_N \underbrace{e^{iL(\mathbf{x}_N,\dot{\mathbf{x}}_N)\delta t} ... e^{iL(\mathbf{x}_1,\dot{\mathbf{x}}_1)\delta t}}_{=\exp i\int dt L} = \int_{\mathbf{x}(t_i)=\mathbf{x}_i}^{\mathbf{x}(t_f)=\mathbf{x}_f} \mathcal{D}\mathbf{x}(t)\, e^{iS}\,, \tag{13}$$

just as claimed. □

1.2 Motivation in field theory

We now turn to states with infinitely many degrees of freedom, instead of just one (or rather three in 3D), but otherwise we follow exactly the same approach as in the QM case. In other words, we consider the following states in the Schrödinger picture:

$$|\mathbf{x}\rangle \text{ with } \hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle \quad \longrightarrow \quad |\phi\rangle \text{ with } \hat{\phi}(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle \tag{14}$$

$$|\mathbf{p}\rangle \text{ with } \hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \quad \longrightarrow \quad |\pi\rangle \text{ with } \hat{\pi}(\mathbf{x})|\pi\rangle = \pi(\mathbf{x})|\pi\rangle\,. \tag{15}$$

Basic properties such as for contractions thus readily generalize from three to infinitely many dimensions, for example

$$\langle \mathbf{p}|\mathbf{x}\rangle = e^{-i\mathbf{p}\cdot\mathbf{x}} \quad \longrightarrow \quad \langle \pi|\phi\rangle = e^{-i\int d^3x\, \pi(\mathbf{x})\phi(\mathbf{x})} \tag{16}$$

$$\langle \mathbf{x}|\mathbf{x}'\rangle = \int \frac{d^3p}{(2\pi)^3}\langle \mathbf{x}|\mathbf{p}\rangle\langle \mathbf{p}|\mathbf{x}'\rangle = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')} = \delta^{(3)}(\mathbf{x}-\mathbf{x}')$$

$$\longrightarrow \quad \langle \phi|\phi'\rangle = \int \mathcal{D}\pi\, \langle \phi|\pi\rangle\langle \pi|\phi'\rangle = \int \mathcal{D}\pi\, e^{i\int d^3x\, \pi(\mathbf{x})\{\phi(\mathbf{x})-\phi'(\mathbf{x})\}} \equiv \delta(\phi-\phi')\,, \tag{17}$$

with the final expression denoting a formally infinite-dimensional delta function.

Let us now consider for simplicity a real scalar field,

$$\hat{\mathcal{H}} = \frac{1}{2}\hat{\pi}^2 + V(\hat{\phi})\,. \tag{18}$$

Deriving the **vacuum transition amplitude** between times t_i and t_f then proceeds along exactly the same lines as the derivation of Eq. (5) – just using, as indicated in Eqs. (16, 17), in each step the infinite-dimensional analogues of the corresponding expressions. The result of this procedure is

$$\langle 0;t_f|0;t_i\rangle \propto \int \mathcal{D}\phi e^{iS}\,, \tag{19}$$

where $S = \int_{t_i}^{t_f}\int d^3x\, \mathcal{L}$.

'Proof'. We literally use the same steps as those leading to Eq. 7, just replacing 3D integrals with infinite-dimensional integrals when inserting N identity factors:

$$\langle 0;t_f|0;t_i\rangle = \langle 0|U(t_f,t_i)|0\rangle = \int \mathcal{D}\phi_1 ... \mathcal{D}\phi_N\, \langle 0|e^{-i\hat{H}(t_N)\delta t}|\phi_N\rangle\langle\phi_N|...\langle\phi_1|e^{-i\hat{H}(t_i)\delta t}|0\rangle\,. \tag{20}$$

The same happens for the infinitesimal expectation values that we need to compute:

$$\langle\phi_{k+1}|e^{-i\hat{H}(t_k)\delta t}|\phi_k\rangle \overset{cf.\ (9)}{=} \int\mathcal{D}\pi\,\langle\phi_{k+1}|\pi\rangle\langle\pi|e^{-i\delta t\int d^3x\,\{\frac{1}{2}\hat{\pi}^2+V(\hat{\phi})\}}|\phi_k\rangle \tag{21}$$

$$\overset{cf.\ (10)}{=} \int\mathcal{D}\pi e^{i\int d^3x\,\pi(\phi_{k+1}-\phi_k)}e^{-i\delta t\int d^3x\,\{\frac{1}{2}\pi^2+V(\phi_k)\}} \tag{22}$$

$$\overset{cf.\ (12)}{\propto} e^{i\delta t\int d^3x\,\mathcal{L}[\phi_k,\dot{\phi}_k]}\,. \tag{23}$$

As before, the product of many of these exponentials can be written as the exponential of a time-integral over $\mathcal{L}$ – thus proving the claim. Note that the ϕ_k correspond to operators in the Schrödinger picture, and thus do not have any time-dependence; the final result, however, really is a path integral over the full $\phi = \phi(t,\mathbf{x})$. □

As a next step, we can ask what happens when inserting factors of field values at specific space-time points on the r.h.s. of Eq. (19). Starting with two field values for example, and by working backwards in the derivation of that equation, it is relatively straightforward to find

$$\int\mathcal{D}\phi\,\phi(x_m)\phi(x_n)\,e^{iS} \propto \langle 0|T\{\hat{\phi}(x_m)\hat{\phi}(x_n)\}|0\rangle\,. \tag{24}$$

'Proof'. Working backwards from Eq. (23) to (20), we directly find

$$\int\mathcal{D}\phi\,\phi(x_m)\phi(x_n)\,e^{iS} \propto \int\mathcal{D}\phi_1...\mathcal{D}\phi_N\,\langle 0|e^{-i\hat{H}(t_N)\delta t}|\phi_N\rangle\langle\phi_N|...\langle\phi_1|e^{-i\hat{H}(t_i)\delta t}|0\rangle\phi(x_m)\phi(x_n)\,. \tag{25}$$

Recall that the field $\phi_k = \phi_k(\mathbf{x})$ is the eigenvalue of $\hat{\phi}$ for a state $|\phi_k\rangle$ in the Schrödinger picture at time t_k. The factor of $\phi(x_m)\phi(x_n) = \phi(t_m,\mathbf{x}_m)\phi(t_n,\mathbf{x}_n)$ thus only affects the path integrals $\mathcal{D}\phi_k$ with $k=m$ or $k=n$. The other integrals we can collapse, using $\mathcal{D}\phi_k|\phi_k\rangle\langle\phi_k| = \mathbb{1}$, leading to

$$\int\mathcal{D}\phi\,\phi(x_m)\phi(x_n)\,e^{iS} \propto \int\mathcal{D}\phi_m\mathcal{D}\phi_n\,\langle 0|U(t_f,t_m)|\phi_m\rangle\langle\phi_m|U(t_m,t_n)|\phi_n\rangle\langle\phi_n|U(t_n,t_i)|0\rangle\phi(x_m)\phi(x_n)\,. \tag{26}$$

Note that $t_1 < ... < t_N$ by construction, so the above expression is only valid if we assume $t_m > t_n$. In the opposite case, for $t_n > t_m$, we'd get

$$\int\mathcal{D}\phi\,\phi(x_m)\phi(x_n)\,e^{iS} \propto \int\mathcal{D}\phi_m\mathcal{D}\phi_n\,\langle 0|U(t_f,t_n)|\phi_n\rangle\langle\phi_n|U(t_n,t_m)|\phi_m\rangle\langle\phi_m|U(t_m,t_i)|0\rangle\phi(x_m)\phi(x_n)\,.$$

This is the basic reason why *path integrals automatically implement time-ordering*. The remaining integrals can be evaluated using

$$\int\mathcal{D}\phi_m\,|\phi_m\rangle\langle\phi_m|\,\phi(x_m) = \hat{\phi}(x_m)\int\mathcal{D}\phi_m\,|\phi_m\rangle\langle\phi_m| = \hat{\phi}(x_m)\,. \tag{27}$$

We thus find

$$\int\mathcal{D}\phi\,\phi(x_m)\phi(x_n)\,e^{iS} \propto \langle 0|U(t_f,t_m)\hat{\phi}(x_m)U(t_m,t_n)\hat{\phi}(x_n)U(t_n,t_i)|0\rangle \tag{28}$$

if $t_m > t_n$. For $t_n > t_m$, on the other hand, we'd have to exchange $m \leftrightarrow n$ on the r.h.s. of this expression. Using further that $t_i < T_n, t_m$ and $t_f > t_n, t_m$, *both* cases reduce to the expression stated in Eq. (24). □

Extending this procedure to more powers of ϕ, and correctly normalizing the vacuum transition amplitude as $\langle 0|0\rangle = 1$ then directly leads to the master formula claimed in Eq. (1). Let us conclude this section by some final remarks:

- The above heuristic derivation explicitly used the simplest possibility of real scalar fields – but can be extended to other cases.
- Similarly, the derivation works in the same way for the full, interacting vacuum $|\Omega\rangle$ as for the free-theory vacuum $|0\rangle$ used above.
- Starting with a Hamiltonian description, and then effectively implementing an inverse Legendre transformation, requires the steps $\mathcal{L} \to \mathcal{H} \to \mathcal{L}$. It is worth mentioning that this is not always possible, or can be difficult to implement in practice. The path integral approach, instead, *defines* a QFT in terms of the (classsical) action, without the need of constructing a Hamiltonian.

2 Calculus

2.1 Gaussian (path) integrals

The only integrals that we typically need to consider explicitly in the path integral formalism are higher-dimensional versions of the fundamental Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \qquad \text{for } a > 0\,. \tag{29}$$

The n-dim. generalization of this identity involves a non-degenerate, symmetric real $n \times n$ matrix $\mathbf{A}$ with positive eigenvalues, and an n-dimensional real vector $\mathbf{J}$:

$$\int d^n x\, e^{-\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{x}\cdot\mathbf{J}} = \sqrt{\frac{(2\pi)^n}{\det \mathbf{A}}}\, e^{\frac{1}{2}\mathbf{J}^T\mathbf{A}^{-1}\mathbf{J}}\,. \tag{30}$$

The proof is left as an exercise – it involves completing the square, diagonalizing $\mathbf{A}$, and then using Eq. (29) n times in this new basis. By analytic continuation the same expression also holds for *complex values* of $\mathbf{x}$ and $\mathbf{J}$ – as long as $\mathbf{A}$ is hermitian and its eigenvalues have positive real parts.

Taking $n \to \infty$, this identity becomes (for real scalar fields ϕ in Minkowski space)

$$\int \mathcal{D}\phi\, e^{-\frac{1}{2}\int d^4x d^4y\, \phi(x)D(x,y)\phi(y)+\int d^4x\, \phi(x)J(x)} \equiv \int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi_x D_{xy}\phi_y+\phi_x J_x} \propto e^{\frac{1}{2}J_x D^{-1}_{xy} J_y}\,. \tag{31}$$

Here, $D(x,y)$ is an arbitrary differential operator, and the very useful shorthand notation that we have introduced in the first step stresses the analogy with Eq. (30). In the final step we discarded a formally infinite prefactor – including a so-called *functional determinant* – because we will soon notice that it eventually cancels whenever we use the above expression above to compute correlation functions as in Eq. (1).

2.2 Generating functionals

For a real scalar field, the generating functional is defined as

$$Z[J] \equiv \int \mathcal{D}\phi\, e^{iS+iJ_z\Phi_z}\,, \tag{32}$$

using the same short-hand notation as above, i.e. $J_z\Phi_z \equiv \int d^4z\, J(z)\Phi(z)$. The space-time function J is known as **source field** (or *source current*).[4] Taking the functional derivative w.r.t. to the source field gives

$$\frac{\delta Z[J]}{\delta J(x)} = i\int \mathcal{D}\phi\, \phi(x) e^{iS+iJ_z\Phi_z}\,, \tag{33}$$

which allows us to re-write our master formula for correlation functions, Eq. (1), as

$$\langle\Omega|T\left\{\phi(x_1)...\phi(x_n)\right\}|\Omega\rangle = \frac{1}{Z[0]}\left(-i\frac{\delta}{\delta J(x_1)}\right)...\left(-i\frac{\delta}{\delta J(x_n)}\right) Z[J]_{J=0}\,. \tag{34}$$

[4] Here you may think of J as a purely auxiliary function, i.e. just a mathematical tool. However, a *classical source* J – 'the right-hand side of the equation of motion', as e.g. in $(\Box + m^2)\Phi(x) = J(x)$ – would also show up in exactly this way in the action. In theories with gauge symmetries, moreover, this is how the conserved *Noether current* ('J') couples to the gauge fields ('ϕ').

The reason that this way of re-writing is extremely useful in practice is that one can compute the path integral in Eq. (32) *explicitly*, albeit *only for the free theory*. To do so, we split $S = S_0 + S_{\rm int}$, where S_0 is always (not just for scalars) quadratic in the fields and describes the free theory, while $S_{\rm int}$ describes the interactions. We can then define

$$Z_0[J] \equiv \int \mathcal{D}\phi \, e^{iS_0 + iJ_z\Phi_z} \,, \tag{35}$$

For a free scalar field, we have

$$S_0 = -\frac{1}{2}\int d^4x \, \phi(x) \left(\Box + m^2 + i\epsilon\right) \phi(x) = -\frac{1}{2}\phi_y \underbrace{\delta_{yx} \left(\Box_x + m^2 + i\epsilon\right)}_{\equiv -iD_{xy}} \phi_x \,, \tag{36}$$

where we added the $i\epsilon$, with $\epsilon > 0$, to make sure the integral in Eq. (35) actually converges. We notice that the inverse of D_{xy}, as introduced in the final step, is just the Feynman propagator D^F:

$$i\left(\Box_x + m^2 + i\epsilon\right) D^F(x,y) = \delta^{(4)}(x-y) \,. \tag{37}$$

Furthermore, Eq. (36) is now of the same form as in Eq. (31). Hence, we get $Z_0 \propto e^{-\frac{1}{2}J_x D^F_{xy} J_y}$, and from Eq. (1) for the free theory we arrive at

$$\langle 0|T\left\{\phi(x_1)...\phi(x_n)\right\}|0\rangle = (-i)^n \frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)} \, e^{-\frac{1}{2}J_x D^F_{xy} J_y}\Big|_{J=0} \,. \tag{38}$$

The above construction works not only for real scalar fields. In summary, and following the same steps as above, the generating functionals for the free theory of real scalars ϕ, complex scalars Φ, vector fields B^μ and fermions ψ are given by

$$Z_0[J] \equiv \int \mathcal{D}\phi \, e^{iS_0 + iJ_z\Phi_z} \propto e^{-\frac{1}{2}J_x D^F_{xy} J_y} \tag{39}$$

$$Z_0[J, J^*] \equiv \int \mathcal{D}\Phi\mathcal{D}\Phi^* \, e^{iS_0 + iJ_z\Phi_z + iJ^*_z\Phi^*_z} \propto e^{-J^*_x D^F_{xy} J_y} \tag{40}$$

$$Z_0[J^\mu] \equiv \int \mathcal{D}B \, e^{iS_0 + i(J_\mu)_z B^\mu_z} \propto e^{-\frac{1}{2}J^\mu_x (D^F_{\mu\nu})_{xy} J^\nu_y} \tag{41}$$

$$Z_0[\eta, \bar\eta] \equiv \int \mathcal{D}\Phi\mathcal{D}\Phi^* \, e^{iS_0 + i\bar\eta_z\psi_z + i\bar\psi_z\eta_z} \propto e^{-\bar\eta_x S^F_{xy}\eta_y} \,. \tag{42}$$

In other words, source fields must always be chosen such that they combine with the respective classical field to a real Lorentz scalar.

2.3 Feynman rules

Feynman rules follow in the usual way from n-point functions, so we just need to use the updated form of our master formula, Eq. (34), to derive them. For the free theory, this simplifies to using Eq. (38) – and corresponding expressions for other fields than

real scalar, cf. the set of equations listed right afterwards. It is very instructive to check explicitly that this gives for example

$$\begin{aligned}
\langle 0|T\left\{\phi(x_1)\phi(x_2)\right\}|0\rangle &= D^F(x_2,x_1)\,, && (43)\\
\langle 0|T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\right\}|0\rangle &= 0\,, && (44)\\
\langle 0|T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right\}|0\rangle &= D^F(x_2,x_1)D^F(x_4,x_3)+ && (45)\\
& \quad D^F(x_3,x_1)D^F(x_4,x_2)+D^F(x_4,x_1)D^F(x_3,x_2)\,,
\end{aligned}$$

just as expected from Wick's theorem.

For the full, interacting theory using Eq. (34) in practice typically involves splitting $S = S_0 + S_{\rm int}$ and expanding the generating functional in the interaction terms (thus relying on perturbative couplings):

$$\begin{aligned}
Z[J] = \int \mathcal{D}\phi\, e^{iS_0+iS_{\rm int}+iJ_z\Phi_z} &= \int \mathcal{D}\phi\, e^{iS_0+iJ_z\Phi_z}\left(1+iS_{\rm int}[\phi]+\frac{1}{2!}(iS_{\rm int}[\phi])^2+...\right) && (46)\\
&= \left(1+iS_{\rm int}\left[-i\frac{\delta}{\delta\phi}\right]+\frac{1}{2!}(iS_{\rm int}\left[-i\frac{\delta}{\delta\phi}\right])^2+...\right)\underbrace{\int \mathcal{D}\phi\, e^{iS_0+iJ_z\Phi_z}}_{Z_0[J]}\,.
\end{aligned}$$

What is meant by the notation in the second line is that for an interaction term of the form $S_{\rm int}[\phi] = \lambda\int d^4z\,\phi(z)^n \equiv \lambda\phi_z^n$ we should use $S_{\rm int}[-i\delta/\delta\phi] = \lambda\int d^4z\,\left(-i\delta/\delta\phi(z)\right)^n \equiv \lambda\left(-i\delta/\delta\phi_z\right)^n$. Using again the result (39) for the free generating functional, the master formula to be used in practice thus becomes

$$\begin{aligned}
&\langle\Omega|T\left\{\phi(x_1)...\phi(x_n)\right\}|\Omega\rangle && (47)\\
&\quad = (-i)^n \frac{\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\left(1+iS_{\rm int}\left[-i\frac{\delta}{\delta\phi}\right]+\frac{1}{2!}(iS_{\rm int}\left[-i\frac{\delta}{\delta\phi}\right])^2+...\right)e^{-\frac{1}{2}J_xD^F_{xy}J_y}\Big|_{J=0}}{\left(1+iS_{\rm int}\left[-i\frac{\delta}{\delta\phi}\right]+\frac{1}{2!}(iS_{\rm int}\left[-i\frac{\delta}{\delta\phi}\right])^2+...\right)e^{-\frac{1}{2}J_vD^F_{vw}J_w}\Big|_{J=0}}\,.
\end{aligned}$$

In order to derive Feynman-rules for **vertices**, we only need the contribution that corresponds to a fully connected diagram at first order, i.e. the term in the above expression that is proportional to $G^F_{x_1z}...G^F_{x_nz}$ (with z being the 'internal' space-time coordinate that is integrated over in $S = \int d^4z\,\mathcal{L}$). In fact, the Feynman rule for a vertex is by definition simply the factor multiplying this product of propagators connecting z with the external points $x_1,...,x_n$. In other words, we only need to consider

$$\langle\Omega|T\left\{\phi(x_1)...\phi(x_n)\right\}|\Omega\rangle \supset i(-i)^n\frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)}S_{\rm int}\left[-i\frac{\delta}{\delta\phi}\right]\,e^{-\frac{1}{2}J_xD^F_{xy}J_y}\Big|_{J=0}\,. \quad (48)$$

Insisting on keeping only terms $\propto G^F_{x_1z}...G^F_{x_nz}$ implies that we only get non-zero contributions from interactions of the form $S_{\rm int}[\phi] = \lambda\phi_z^m$ for $n = m$. There are then $n!$ options that yield the relevant 'contractions', resulting in the **Feynman rule** being $i\lambda\, n!$ (it is very instructive to convince yourself that this statement is *exactly* correct, including signs and further factors of i !). Corresponding considerations are valid for other types of fields, not just the real scalars ϕ used here as example.

A Functionals for ~~dummies~~ physicists

For all practical purposes in physics, a **function** is a (sufficiently often differentiable) *map*

$$f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m \tag{49}$$

that assigns a value $f(x) \in \mathbb{R}^m$ to an input $x \in \mathbb{R}^n$. A **functional**, instead, is a map that takes such a function f as input and returns a (typically real) number $F[f]$:

$$F : D(\Omega) \longrightarrow \mathbb{R}\,. \tag{50}$$

Here, $D(\Omega)$ denotes the 'space of all test functions over Ω'. In practice, this just means that $f \in D(\Omega)$ must be sufficiently 'nice' (e.g. often enough differentiable) for the functional to be defined.

Mathematically, a **distribution** is defined as a continuous, linear functional. In physics, we refer by this name to what is known as a *regular distribution* in math, which has the additional property that the functional F can be fully represented by some locally integrable function ϕ:

$$F[f] = \int_\Omega dx\, \phi(x) f(x) \qquad \text{for } \textit{all } f \in D(\Omega)\,, \tag{51}$$

where the '*all* ' in the definition is most decisive. In physics, such (regular) distributions are essentially the only functionals that we are interested in.[5]

We can next introduce the **functional differential** of a functional $F[f]$ – also known as its *first variation in the direction* φ – which itself is a (not necessarily regular) distribution defined as

$$\delta F[f, \varphi] \equiv \lim_{\epsilon \to 0} \frac{F[f + \epsilon\varphi] - F[f]}{\epsilon}\,, \tag{55}$$

[5]The Dirac delta 'function' $\delta(x)$ is a notable exception, which however for notational ease is in practice still used as if it *was* regular:

$$\delta[f] \equiv f(0) \text{ '=' } \int dx\, \delta(x) f(x)\,. \tag{52}$$

Here, the first step introduces the Dirac *distribution* in a mathematically fully well-defined way; the second step *pretends* that there exists a function $\delta(x)$ with the required property for *any* function $f \in D(\Omega)$. Such a function, however, cannot exist – i.e. the Dirac distribution is not actually regular. One way of seeing this is for example by looking at a commonly adopted possible representation of the Dirac functional in terms of a series of functions $\delta_a(x) \equiv (2\pi a)^{-1/2} \exp(-x^2/2a^2)$:

$$\delta[f] = \lim_{a \to 0} \int dx\, \delta_a(x) f(x)\,, \tag{53}$$

which correctly results in $\delta[f] = f(0)$ for 'any' f. One might then be tempted to think that $\delta(x) = \lim_{a\to 0} f_a(x)$ could work, but this would in fact result in

$$\int dx\, \delta(x) f(x) = \int dx\, f(x) \lim_{a \to 0} f_a(x) = \int dx\, f(x) \times \left\{ \begin{array}{ll} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{array} \right\} = 0\,. \tag{54}$$

where $f, \varphi \in D(\Omega)$. In situations of relevance in physics, the functional differential typically has an integral representation w.r.t. its second argument (the 'direction' in function space):

$$\delta F[f,\varphi] = \int_\Omega dx\, \frac{\delta F}{\delta f}(x)\varphi(x) \qquad \text{for } \textit{all } \varphi \in D(\Omega)\,. \tag{56}$$

Here, the function $\delta F/\delta f$ takes the same role as the function ϕ in Eq. (51) and is known as the **functional derivative**.

For the purpose of practical applications, the calculus with functional derivatives greatly simplifies by noting that any *function* $f : \mathbb{R}^n \to \mathbb{R}$ trivially induces the *functional* $F_y : f \in D(\Omega) \to F_y[f] \equiv f(y) \in \mathbb{R}$. Using Eq. (55), it is then straightforward to show that $\delta F_y[f,\varphi] = \varphi(y)$. Comparison with Eq. (56) thus implies that the functional derivative of F_y is the Dirac delta function:

$$\frac{\delta F_y}{\delta f}(x) = \delta(x-y) \equiv \frac{\delta f(y)}{\delta f(x)}\,, \tag{57}$$

where in the last step we have introduced the more commonly used notation for this statement. All of this can be summarized by the following **basic axiom** of functional differentiation – which in practice is the only thing you need to know for practical applications in Minkowski space:[6]

$$\frac{\delta}{\delta f(x)} f(y) \equiv \delta^{(4)}(x-y) \qquad \Leftrightarrow \qquad \frac{\delta}{\delta f(x)} \int d^4y\, f(y)\varphi(y) \equiv \varphi(x)\,. \tag{59}$$

As expected from this definition, functional derivatives behave to a large degree as ordinary derivatives. In particular, the following identities hold, which are very useful from a practical point of view:

$$\frac{\delta}{\delta f(x)}\big(g(y)h(y)\big) = g(y)\frac{\delta h(y)}{\delta f(x)} + h(y)\frac{\delta g(y)}{\delta f(x)} \tag{60}$$

$$\frac{\delta}{\delta f(x)}\, g\left(\int d^4y\, f(y)\varphi(y)\right) = \phi(x)\, g'\left(\int d^4y\, f(y)\varphi(y)\right) \tag{61}$$

$$\frac{\delta}{\delta f(x)} \int d^4y\, \big(\partial_\mu f(y)\big)\varphi^\mu(y) = -\partial_\mu \varphi^\mu(x)\,. \tag{62}$$

Here, the first two equalities can be easily understood from analogies as pointed out in footnote 6, while the latter follows after partial integration. (Note that the arbitrary functions g and h in the first equality should have an explicit functional dependence on f, not just depend on the argument y, otherwise the respective functional derivatives $\delta/\delta f$ will vanish identically).

[6] For the purpose of a more intuitive understanding of functional expressions, furthermore, it can sometimes be helpful to explicitly consider their respective finite-dimensional versions. Representing the continuous functions $f(x)$ and $\phi(x)$ by discrete values f_i and ϕ_i, e.g., the basic axiom just takes the much more familiar form

$$\frac{\partial}{\partial f_i} f_j = \delta_{ij} \qquad \Leftrightarrow \qquad \frac{\partial}{\partial f_i}(f_j\phi_j) = \phi_i\,. \tag{58}$$