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FYS5120: Preliminaries

FYS5120 draws heavily on results developed in QFT I (FYS4170), both concerning conceptual understanding (of the quantization procedure and perturbation theory) and practical familiarity with rate calculations (for which some additional background from classes in particle physics is typically very helpful). These notes summarize some of the *main concepts and results that you absolutely need to know* from before; if any of this sounds even slightly unfamiliar, make sure to fill in the gaps asap (and in any case before the end of January). The notes also fix our notation and conventions (following those of Peskin & Schroeder, PS). We will, e.g., use natural units where $\hbar = c = 1$ (though these constants may sometimes be kept for pedagogic reasons).

1 Canonical quantization

1.1 From non-relativistic particles to quantum mechanics

A **classical** system of N particles is fully determined by its generalized coordinates q_a and canonically conjugated momenta $p_a \equiv \partial L/\partial \dot{q}_a$, where $a \in \{1, 3N\}$ and $L = L(q, \dot{q})$ is the *Lagrange function* that describes the system. The least-action principle, in particular, leads to the **Euler-Lagrange equations** of motion:

$$\frac{dL}{dq_a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} = 0\,. \tag{1}$$

Alternatively, a Legendre transformation allows (if $\det \partial^2 L/\partial \dot{q}_a \partial q_b \neq 0$) to replace the Lagrangian L with the *Hamiltonian* H which no longer depends on the velocities:

$$H(p, q) \equiv p_a \dot{q}_a(p, q) - L[q, \dot{q}(p, q)]\,. \tag{2}$$

In this formulation, the system is described by the **Hamilton equations** of motion,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \tag{3}$$

i.e. $2 \times 3N$ coupled first-order ODEs instead of the $3N$ second-order ODEs in Eq. (1). Notably, the time evolution of any observable $f = f(t, q, p)$ thus becomes

$$\frac{df}{dt} = \partial_t f - \{H, f\}\,, \tag{4}$$

where the explicit time-dependence of f – and hence the ∂_t term – never appears in closed systems (which covers all applications relevant to our context). Recall that the Poisson bracket of any two phase-space functions $f(q, p)$ and $g(q, p)$ is defined as

$$\{f, g\} \equiv \frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q_a}\,, \tag{5}$$

which directly implies

$$\{q_a, p_b\} = \delta_{ab} \qquad \text{and} \qquad \{q_a, q_b\} = \{p_a, p_b\} = 0\,. \tag{6}$$

A **state** in classical mechanics is simply a point (p, q) in phase space, and any phase-space function $f(p, q)$ is a (potential) observable. In quantum mechanics, this is very different. Here the state of a system is described by a vector $|\psi\rangle$ in Hilbert space, and possible observations correspond to the *eigenvalues* $\mathcal{O}_n$ of Hermitian operators ('observables') $\hat{\mathcal{O}}$. The *expectation value* of $\hat{\mathcal{O}}$ – i.e. the observation that is expected, on average, for a given state $|\psi\rangle$ – is $\langle\psi|\hat{\mathcal{O}}|\psi\rangle = \sum_n \mathcal{O}_n \, |\langle\psi|n\rangle|^2$, where $|n\rangle$ is the eigenstate corresponding to the eigenvalue $\mathcal{O}_n$. Any given measurement, however, will always produce a value $\mathcal{O}_n$ (with probability $|\langle\psi|n\rangle|^2$); in the standard interpretation, this is accompanied by the 'collapse' of the state $|\psi\rangle$ to the state $|n\rangle$.

The **canonical quantization** of a classical system consists in promoting phase-space functions $f(q, p)$ to operators $\hat{f} = f(\hat{q}, \hat{p})$ that act on quantum states $|\psi\rangle$. These operators are constructed such that they obey commutation relations instead of the classical Poisson brackets:

$$\{\,,\,\}_{\text{classical}} \to -\frac{i}{\hbar}\,[\,,\,]_{\text{quantum}}\,. \tag{7}$$

For example, this immediately gives

$$[\hat{q}_a, \hat{p}_b] = i\hbar\,\delta_{ab} \qquad \text{and} \qquad [\hat{q}_a, \hat{q}_b] = [\hat{p}_a, \hat{p}_b] = 0\,. \tag{8}$$

In fact, Eqns. (7) and (8) are equivalent (up to the first bullet point below) because any observable can be written as a function of q and p. Importantly, this quantization prescription

- is not unique due to *ordering ambiguities*, i.e. there are in general several operators $\hat{f}$ that correspond to a given classical function f.
- assumes that all operators are evaluated at equal times, i.e. it assumes the *Schrödinger picture*.
- is only viable for *non-relativistic systems*. Quantizing a non-relativistic Hamiltonian in the same way would e.g. violate both *i)* causality and *ii)* the conservation of probability, *iii)* allow for Hamiltonians not bounded from below, and *iv)* imply that the number of particles is always conserved, in sharp contrast to the equivalence of rest mass and kinetic energy required by special relativity.

1.2 From (integer spin) classical fields to quantum field theory

While relativistic *particles* cannot be consistently quantized, relativistic **fields** can. Here the field values *at each space-time point* then take the role of the generalized coordinates, $q_a \to \phi(x^\mu)$, and all the physics is encoded in a *Lagrangian density* $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ – which often is simply referred to as 'Langrangian'. Extremizing the action $S = \int d^4x\,\mathcal{L}$ then again leads to **Euler-Lagrange** equations:[1]

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0\,. \tag{9}$$

[1] NB: ϕ can also carry spacetime or internal indices. For a vector field, e.g., simply replace $\phi \to A^\nu$.

We can define a *canonical momentum density* $\pi(x) \equiv \partial\mathcal{L}/\partial\dot{\phi}$, and a *Hamiltonian density* $\mathcal{H} \equiv \pi\dot{\phi} - \mathcal{L}$. Note that $\mathcal{H} = \mathcal{H}(\phi, \pi, \nabla\phi)$; the additional dependence on $\nabla\phi$ implies that the Hamilton equations of motions for fields are much more complicated than in the classical mechanics case, and therefore less freqently used in practice, though Poisson brackets work in similar ways. Via Noether's theorem, $\mathcal{H}$ and π as just defined are related to the *physical* energy and momentum of the system as

$$H = \int d^3x \left(\pi\dot{\phi} - \mathcal{L}\right), \tag{10}$$

$$\mathbf{P} = -\int d^3x \pi\nabla\phi\,. \tag{11}$$

Compared to the classical mechanics case above, we can thus identify two important (and in fact related) differences:

- The equations of motion are automatically, and manifestly, Lorentz invariant
- Classical fields can be described as the limit $N \to \infty$ of classical mechanics. While for the latter, N is constant and hence necessarily conserved, a theory with *infinitely* many degrees of freedom does not suffer from this restriction.

These two aspects successfully address the worrisome third bullet point in the list after Eq. (8), allowing the successful **canonical quantization** of classical fields by promoting them to *field operators*. Keeping in mind the analogy of $q_a \simeq \phi(x^\mu)$ and $p_a \simeq \pi(x^\mu)$, Eq. (8) becomes[2]

$$[\hat{\phi}(t_0, \mathbf{x}), \hat{\pi}(t_0, \mathbf{y})] = i\,\delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{12}$$

$$[\hat{\phi}(t_0, \mathbf{x}), \hat{\phi}(t_0, \mathbf{y})] = [\hat{\pi}(t_0, \mathbf{x}), \hat{\pi}(t_0, \mathbf{y})] = 0\,. \tag{13}$$

Note that we are still in the Schrödinger picture, i.e. the field operators $\hat{\phi}$ and $\hat{\pi}$ do not have any time-dependence; instead, as indicated, the commutator relations must be specified at some common time t_0. Instead of working with the operators $\hat{\phi}$ and $\hat{\pi}$, we can change the basis by defining new operators $\hat{a}$ and $\hat{b}$ through

$$\hat{\phi}(\mathbf{x}) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\hat{a}_\mathbf{p} e^{i\mathbf{px}} + \hat{b}^\dagger_\mathbf{p} e^{-i\mathbf{px}}\right), \tag{14}$$

where $\omega_p \equiv +\sqrt{\mathbf{p}^2 + m^2}$ and m is the mass of ϕ. This *implies*[3]

$$\hat{\pi}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \frac{\sqrt{\omega_p}}{2} \left(\hat{a}_\mathbf{p} e^{i\mathbf{px}} - \hat{b}^\dagger_\mathbf{p} e^{i\mathbf{px}}\right). \tag{15}$$

[2] In principle, we also need the operator-valued version of the independent degrees of freedom represented by $\nabla\phi$; for this, we can simply use $\nabla\hat{\phi}$.

[3] This is because $\phi(x^\mu)$ must solve the Klein-Gordon equation, which is only possible if the $e^{\pm i\mathbf{px}}$ in Eq. (14) appear as parts of plane-wave factors $e^{\mp ip\cdot x}$ with $p^0 = \omega_p$. Using $\pi = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}$ then gives the claimed expression, after absorbing the same constant phase $e^{\mp it_0\omega_p}$ into the def. of $\hat{a}, \hat{b}$. Note that the last equality only holds in the free theory.

In the above expressions, $a = b$ for *real* (classical) fields. Eqs. (12,13) result in corresponding commutation relations for the ladder operators,

$$\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{k}}\right] = \left[\hat{b}_{\mathbf{p}}, \hat{b}^{\dagger}_{\mathbf{k}}\right] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{k}) \,, \tag{16}$$

with all other commutators vanishing. These relations can be seen as an alternative (and fully equivalent) way of stating canonical quantization.

The **interpretation** of the quantized fields is greatly helped by the observation that the ladder operators[4] a and b *diagonalize* both energy and momentum operators of the system. Namely, plugging Eqs. (14,15) into Eqs. (10,11) gives[5]

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a^{\dagger}_{\mathbf{p}} a_{\mathbf{p}} + b^{\dagger}_{\mathbf{p}} b_{\mathbf{p}}\right) \,, \tag{17}$$

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a^{\dagger}_{\mathbf{p}} a_{\mathbf{p}} + b^{\dagger}_{\mathbf{p}} b_{\mathbf{p}}\right) \,. \tag{18}$$

This result lets us draw two important conclusions:

1. The Hamiltonian is bounded from below. The **vacuum** $|0\rangle$ is defined as the state of lowest energy, which in the case of the above Hamiltonian implies $H|0\rangle = 0$. This leads to the alternative definition of $|0\rangle$ as the state for which $a_{\mathbf{p}}|0\rangle = b_{\mathbf{p}}|0\rangle = 0$, for all $\mathbf{p}$.

2. The state $|\psi\rangle = a^{\dagger}_{\mathbf{p}_1} ... a^{\dagger}_{\mathbf{p}_n}|0\rangle$ is an *eigenstate* of both H and $\hat{\mathbf{P}}$, with *eigenvalues* $E = \omega_{p_1} + ... + \omega_{p_n}$ and $\mathbf{P} = \mathbf{p}_1 + ... + \mathbf{p}_n$, respectively. This is the basis for the **particle** interpretation, which is the name we choose to give to these Lorentz-invariant field excitations. The same holds for $a \to b$, in which case we speak of **anti-particles**.

A convenient definition is that of a one-particle state

$$|\mathbf{p}\rangle \equiv \sqrt{2E_p} a^{\dagger}_{\mathbf{p}}|0\rangle \,, \tag{19}$$

which allows us to write the Hamiltonian in a the manifestly Lorentz-invariant form (up to, as expected, a factor E corresponding to the *physical* energy)

$$H = \int \frac{d^3p}{(2\pi)^3 2E} \, E\big(|\mathbf{p}\rangle\langle\mathbf{p}| + |\mathbf{p}\rangle_c\langle\mathbf{p}|_c\big), \tag{20}$$

where $|\mathbf{p}\rangle_c \equiv \sqrt{2E_p} b^{\dagger}_{\mathbf{p}}|0\rangle$ is a one-*anti*particle state. We can also define the state describing a particle, or antiparticle, at position $\mathbf{x}$:

$$|\mathbf{x}\rangle \equiv \phi^{\dagger}(\mathbf{x})|0\rangle, \qquad |\mathbf{x}\rangle_c \equiv \phi(\mathbf{x})|0\rangle \,, \tag{21}$$

[4]From this point on, we'll happily drop the hat whenever it is more or less clear from the context that a given quantity is an operator.

[5]Up to an infinite constant corresponding to the energy density (and pressure) of the vacuum.

which is a useful definition as it recovers in both cases the result $\langle \mathbf{x}|\mathbf{p}\rangle = {}_c\langle \mathbf{x}|\mathbf{p}\rangle_c = e^{i\mathbf{px}}$ familiar from quantum mechanics.

The final step is to (re-)introduce the **full time dependence** of the field operators. This is achieved by moving from the above discussed field operators in the Schrödinger picture, $\phi(\mathbf{x})$, to time-dependent operators in the Heisenberg picture, $\phi(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt}$. An explicit calculation of this step leads to

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2p^0}}\left(\hat{a}_{\mathbf{p}}e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}}e^{ip\cdot x}\right), \tag{22}$$

where $p_0 = \omega_p > 0$ is a *result* of the entire procedure, not something that is put in by hand. Unlike in the naive approach, cf. section 1.1, the correct quantization prescription thus predicts two different types of quanta (or one, for real fields) with *positive* energies which can be either created (positive *frequency* modes, $\propto e^{-ip\cdot x}$) or destroyed (negative *frequency* modes, $\propto e^{ip\cdot x}$). Eq. (22) also very nicely illustrates the fundamental *particle-wave-duality* of quantum fields: as discussed above, the operators a, b directly point to the particle interpretation; the $e^{\pm ip\cdot x}$, on the other hand, make ϕ a solution to the Klein-Gordon equation, thus pointing to the wave interpretation.

1.3 Quantising half-integer fields

For spin-1/2 fields ψ, we must replace the commutators in Eqs. (12,13) with **anti-commutators**:

$$\{\psi(t_0,\mathbf{x}), \pi(t_0,\mathbf{y})\} = i\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\mathbb{1}\,, \tag{23}$$
$$\{\psi(t_0,\mathbf{x}), \psi(t_0,\mathbf{y})\} = \{\pi(t_0,\mathbf{x}), \pi(t_0,\mathbf{y})\} = 0\,, \tag{24}$$

where now $\pi = \partial\mathcal{L}/\partial\dot{\psi} = i\psi^\dagger$ in the free theory. With this starting point, we can perform the exactly analogous steps as for bosonic particles, ultimately resulting in

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2p^0}}\sum_s \left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\,\dagger}_{\mathbf{p}}v^s e^{ip\cdot x}\right). \tag{25}$$

Here, u^s (v^s) are the positive (negative) frequency solutions of the free Dirac equation, with $s = 1, 2$ denoting the spin direction. The ladder operators obey

$$\left\{a^s_{\mathbf{p}}, a^{r\,\dagger}_{\mathbf{k}}\right\} = \left\{b^s_{\mathbf{p}}, b^{r\,\dagger}_{\mathbf{k}}\right\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\mathbf{p}-\mathbf{k})\,. \tag{26}$$

Similar to before, an anti-particle at position $\mathbf{x}$ is described by the state $\psi(x)|0\rangle$; a particle is described by $\overline{\psi}(x)|0\rangle$.

Just as the commutator relations in Eq. (16) imply that a state may contain an arbitrary number of particles with identical quantum numbers, the anti-commutators of Eq. (26) imply that there can be at most one particle with a given set of quantum numbers. This is expected from Pauli's much more general **spin-statistics theorem**:

1. Particles with integer spin are 'bosons': they obey Bose-Einstein statistics, and must hence be quantized using commutator relations.

2. Particles with half-integer spin are 'fermions': they obey Fermi-Dirac statistics, and must hence be quantized using anti-commutator relations.

Technically, this holds for any possible QFT that *i)* obeys Lorentz invariance, *ii)* has a Hamiltonian bounded from below, *iii)* does not contain negative norm states and *iv)* does not violate causality. In other words: for any reasonable QFT.

2 Observables

The quantization procedure summarized above referred only to the free part of theory, assuming no interactions that would allow to test such a theory. In general any theory can be decomposed as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \tag{27}$$

into a free part ($\mathcal{L}_0$) that is quadratic in the fields, and an interacting part ($\mathcal{L}_{\text{int}}$) that is not.

2.1 Correlation functions

All potential observables of interest can be expressed in terms of **correlation functions**, which thus constitute the basic building block to calculate cross sections, decay rates, etc. In a sense, an algorithm to compute correlation functions can even be viewed as *defining* a given QFT – a notion which we will get back to later in this course. For canonical quantization, this algorithm is summarized in what can be called the **master formula** for an n-point correlation function:

$$\langle\Omega|T\left\{\phi(x_1)...\phi(x_n)\right\}|\Omega\rangle = \lim_{\mathcal{T}\to\infty(1-i\epsilon)} \frac{\left\langle 0\middle|T\left\{\phi_I(x_1)...\phi_I(x_n)\exp\left[-i\int_{-\mathcal{T}}^{\mathcal{T}}dtH_I(t)\right]\right\}\middle|0\right\rangle}{\left\langle 0\middle|T\left\{\exp\left[-i\int_{-\mathcal{T}}^{\mathcal{T}}dtH_I(t)\right]\right\}\middle|0\right\rangle}\,. \tag{28}$$

Here, $|\Omega\rangle$ denotes the vacuum of the full (interacting) theory, and $T\{\}$ indicates time-ordering (i.e. the operators inside the parenthesis are understood to be ordered such that their time arguments increase from right to left). Further, $H_I \equiv H_{\text{int}}[\phi_I]$, where ϕ_I is the field operator in the *interaction picture* in which the time evolution is with H_0 (rather than the full $H = H_0 + H_{\text{int}}$ as it would be the case in the Heisenberg picture). As a consequence, the diagonalization of the Hamiltonian H_I works technically in the same way as for the free theory, by introducing ladder operators with identical properties as discussed above.

Notably, our master formula to compute correlation functions is in principle *exact*. Its real power, however, comes from the fact that it is particularly suitable for **perturbation theory**, where it can be expanded in small couplings. This is because any perturbative interaction will only contain monomials of the form $H_I \supset \frac{\lambda}{m!}\phi_I^m$, with some coupling constant $\lambda \ll 1$. By expanding the exponentials in Eq. (28), it then becomes clear that the entire r.h.s. of that equation can be written as a (typically rather large) sum of terms of the form

$$\left\langle 0\middle|T\left\{\phi_I(x_1)...\phi_I(x_{m'})\right\}\middle|0\right\rangle, \tag{29}$$

with $m' = n + d \cdot m$ (where d is the number of 'vertices'). In other words, n-point functions of the *interacting* theory can be fully expressed in terms of n-point functions of the *free* theory. The latter are very straight-forward to evaluate, using Wick's theorem. The result of the entire procedure (perturbative expansion + evaluation of free-theory correlation functions) can be visualized by the use of **Feynman diagrams**; at the same time, the associated **Feynman rules** provide a very efficient means of evaluating the r.h.s. of Eq. (28).

2.2 Dynamics: the invariant matrix element

The **S-matrix** is the fundamental quantity of interest for observables related to individual particle interactions. It specifies the probability (amplitude) for a transition between arbitrary *'in states'* (prepared as localized wave-packages with momenta $\mathbf{k}_i$ in the infinite past) and *'out states'* (in terms of plane-wave solutions with momenta $\mathbf{p}_f$ in the distant future):

$$ {}_{\text{out}}\langle \mathbf{p}_1 \mathbf{p}_2 ... | \mathbf{k}_1 \mathbf{k}_2 ... \rangle_{\text{in}} \equiv \langle \mathbf{p}_1 \mathbf{p}_2 ... | S | \mathbf{k}_1 \mathbf{k}_2 ... \rangle \,. \tag{30} $$

On the r.h.s. of this definition of S, importantly, the one-particle states $|\mathbf{p}_f\rangle$ and $|\mathbf{k}_i\rangle$ are prepared at the *same* (arbitrary) reference time. The S-matrix splits into a trivial part (no interactions) and the so-called T-matrix which is non-zero iff interactions of the specified type ($|\text{in}\rangle \to |\text{out}\rangle$) are possible:

$$ S = \mathbb{1} + iT \,. \tag{31} $$

Four-momentum conservation requires T to be of the form

$$ \langle \mathbf{p}_1 \mathbf{p}_2 ... | iT | \mathbf{k}_1 \mathbf{k}_2 ... \rangle = (2\pi)^4 \delta^{(4)} \left(\textstyle\sum_i \mathbf{k}_i - \sum_f \mathbf{p}_f \right) \cdot i\mathcal{M}(\mathbf{k}_1 \mathbf{k}_2 ... \to \mathbf{p}_1 \mathbf{p}_2 ...) \,. \tag{32} $$

This equation defines the relativistically **invariant matrix element** $\mathcal{M}$. The LSZ reduction formula states that it is essentially the Fourier transform of a correlation function – which means that we can use very *similar* Feynman rule techniques to calculate it. In particular, only fully connected and amputated diagrams contribute to $\mathcal{M}$. In practice, calculating $\mathcal{M}$ for various processes is part of the daily bread and butter of (theoretical) particle physics; mastering it requires not only theoretical understanding but significant hands-on training and experience.

The invariant matrix element is absolutely crucial as it contains all information about the underlying *dynamics* of a given process, i.e. the effect of the (model-dependent) interactions. It can directly be related to important observables such as the **cross section** σ (for processes with two particles in the initial state) or **decay rate** Γ (for processes with one particle in the initial state):

$$ d\sigma = \frac{1}{4E_A E_B |v_A - v_B|} d\Pi_n \left| \mathcal{M}(\mathbf{k}_A \mathbf{k}_B \to \mathbf{p}_1 ... \mathbf{p}_n) \right|^2 , \tag{33} $$

$$ d\Gamma = \frac{1}{2m_A} d\Pi_n \left| \mathcal{M}(m_A \to \mathbf{p}_1 ... \mathbf{p}_n) \right|^2 , \tag{34} $$

where for definiteness we have considered a fixed number of n particles in the final state. Note that the structure of these (and similar) expressions is always the same: some kinematical factor multiplying $|\mathcal{M}|^2$, summed / integrated over *all possible physical configurations of the final state*. The latter is captured by the Lorentz invariant n-body **phase-space**, which we turn to next.

2.3 Kinematics

The (infinitesimal) phase-space for n particles is defined as

$$d\Pi_n \equiv \prod_{f=1}^{n} \left(\frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \, \delta^{(4)}\big(P - \textstyle\sum_f p_f\big) \,, \tag{35}$$

where $P = \sum_i k_i$. As already described above, this expression can be computed completely independent of the interactions taking place, and is attributed to the *kinematics* rather than the dynamics of a process. In passing, let us note that Π_n has mass dimension $2n-4$. Eqs. (33,34) thus imply that $|\mathcal{M}|^2$ must have mass dimension $4-2n$ $(6-2n)$ for a two-particle (one-particle) initial state.

The most commonly encountered case are two-particle final states, i.e. $n = 2$. There exist various closed-form expressions for Π_2, and it is the specific application that determines which one is most relevant to use. For scattering events in the CMS frame, e.g., it is useful to specify the remaining two integrals in Eq. (35) in terms of the CMS scattering angles:

$$d\Pi_2 = \frac{1}{16\pi^2} \frac{|\mathbf{p}_1^{\rm CM}|}{E_{\rm CM}} d\phi_{\rm CM} \, d\cos\theta_{\rm CM} \,. \tag{36}$$

If $|\mathcal{M}|^2$ is symmetric about the collision axis, e.g. because we averaged over all possible initial-state spin orientations, one of the integrations can be performed trivially by replacing $d\phi_{\rm CM} \to 2\pi$. For the remaining integral, note that $\cos\theta_{\rm CM}$ takes the usual values from -1 to 1 if the final-state particles are different – but only from 0 to 1 if they are identical. This is because θ and $\pi - \theta$ correspond to identical physical configurations in the latter case.

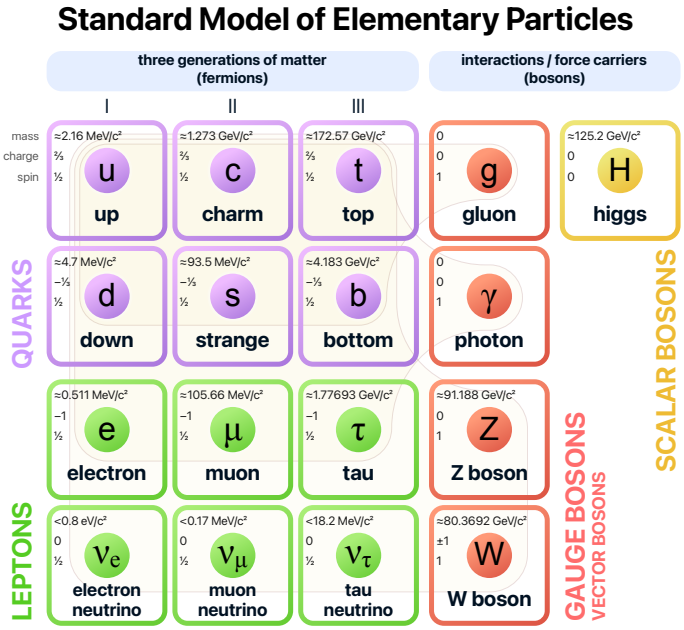
For $2 \to 2$ there are further useful, alternative expressions. One frequent application is to express the scattering angles in the lab frame (where one of the incoming particles is at rest), see e.g. the treatment of Compton scattering in PS. Alternatively there is also a way to express the differential cross section exclusively in terms of Lorentz-invariant quantities,

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s} \frac{1}{|\mathbf{k}_1^{\rm CM}|^2} |\mathcal{M}|^2 \,, \tag{37}$$

where s and t are Mandelstam variables. In both of the above cases, one must again add a symmetry factor of $\frac{1}{2}$ if the two final-state particles are identical.

3 A particle physics 'cheat sheet'

One of the main goals of this course is to understand the fundamental building blocks and theoretical concepts behind the standard model of particle physics (SM). *Formally*, FYS5120 does not require any background in particle physics. In order to keep the focus and not to get lost in the details of theoretical concepts introduced on the way, however, it is very useful to have at least an approximate idea of where we are going, i.e. what kind of thing we want to understand in some more detail. For this purpose the following graphical summary of the SM (from Wikipedia) is very useful:



The list above shows all known *fundamental*, point-like particles. Everything we colloquially refer to as *matter* is made of two types of spin-1/2 fermions, leptons and quarks. **Leptons** comprise the familiar electron and an electrically neutral version of this particle, the neutrino. Both the electron and the neutrino further have two 'cousins' with identical properties except for their masses; we say that there are three *generations* of leptons. **Quarks** come also in two versions, up-type and down-type, and in three generations (with identical properties except for their masses). Protons and neutrons are – at least to first approximation – made of three first-generation quarks. In general, objects containing three quarks (of any generation) are referred to as *baryons*, objects consisting of any quark-antiquark pair as *mesons*. Other composite objects containing quarks are presently not known (though they might exist). Unlike leptons, it is however fundamentally impossible to isolate individual quarks – and we'll understand at the end of the course why exactly that is the case.

Beyond electromagnetism, which is mediated by photons as described in QED, there are two more fundamental forces in nature that govern interactions at the microscopic level: the **weak** and the **strong force**. Just like the photon for electromagnetism, the force mediators of the strong force (g) and weak force ($Z, W^{\pm}$) are vector bosons, i.e. spin-1 particles. If you zoom in, the graphic tells you which partic-

les can 'feel' which of the three fundamental forces (neutrinos, e.g., are neutral under electromagnetism and the strong force – but not under the weak force). The **Higgs boson**, finally, takes a special role not only because it is the only know fundamental particle with spin-0. As we will learn in some detail, it is needed for consistency to explain the masses of all the other SM particles.

Maybe not too surprisingly from a quick glance at the figure, the above is sometimes summarized by stating that 'there are 17 different fundamental particles'. One of the things we will learn in this course is why this statement is highly misleading, not to say outright wrong, and how to correctly count the number of fundamental particles in nature. For now, you should just *know the spins and electric charges of all fundamental particles, and have a (rough) idea about their mass scale.*