Chapter 1

The algebraic origin of SUSY

The goal of these lectures is to introduce the basics of low-energy models of supersymmetry (SUSY) using the Minimal Supersymmetric Standard Model (MSSM) as a main example. Rather than starting with the problems of the SM, we will focus on the algebraic origin of SUSY in the sense of an extension of the symmetries of Einsten's Special Relativity (SR), which was the original motivation for SUSY.

1.1 What is a group?

Definition: The set $G = \{g_i\}$ and operation $\bullet$ form a **group** if and only if for $\forall\, g_i \in G$

i) $g_i \bullet g_j \in G$ (closure)

ii) $(g_i \bullet g_j) \bullet g_k = g_i \bullet (g_j \bullet g_k)$ (associativity)

iii) $\exists e \in G$ such that $g_i \bullet e = e \bullet g_i = g_i$ (identity element)

iv) $\exists g_i^{-1} \in G$ such that $g_i \bullet g_i^{-1} = g_i^{-1} \bullet g_i = e$ (inverse)

A simple example of a group is $G = \mathbb{Z}$ with usual addition as the operation, $e = 0$ and $g^{-1} = -g$. Alternatively we can restrict the group to $\mathbb{Z}_n$, where the operation is addition with modulo n. In this group, $g_i^{-1} = n - g_i$ and the unit element is $e = 0$. Note that $\mathbb{Z}$ is an *infinite* group, while $\mathbb{Z}_n$ is finite, with *order* n (meaning n members). Both are *abelian* groups, meaning that $g_i \bullet g_j = g_j \bullet g_i$.

All of this is "only" mathematics. Physicists are often more interested in groups where the elements of G *act* on some elements of a set $s \in S$, $g(s) = s' \in S$.[1] S here can for example be the state of a system, say a wave-function in quantum mechanics. We will return to this in a moment, let us just mention that the operation $g_i \bullet g_j$ acts as $(g_i \bullet g_j)(s) = g_i \bullet (g_j(s))$ and the identity acts as $e(s) = s$.[2]

[1] As a result mathematics courses in group theory are not always so relevant to a physicist.

[2] We can prove this from iii) in the definition. Note that we use e as the identity in an abstract group, while

A more sophisticated example of a group can be found in a use for the Taylor expansion[3]

$$\begin{aligned} f(x+a) &= f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \dots \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{d^n}{dx^n} f(x) \\ &= e^{a\frac{d}{dx}} f(x) \end{aligned}$$

The operator $T_a = e^{a\frac{d}{dx}}$ is called the **translation operator** (in this case in one dimension). Together with the operation $T_a \bullet T_b = T_{a+b}$ it forms the **translational group** $T(1)$, where $T_a^{-1} = T_{-a}$. In N dimensions the group $T(N)$ has the elements $T_{\vec{a}} = e^{\vec{a}\cdot\vec{\nabla}}$.

Definition: A subset $H \subset G$, is a **subgroup** if and only if:[a]

i) $h_i \bullet h_j \in H$ for $\forall h_i, h_j \in H$

ii) $h_i^{-1} \in H$ for $\forall h_i \in H$

[a] An alternative, more compact, way of writing these two requirements is $h_i \bullet h_j^{-1} \in H$ for $\forall h_i, h_j \in G$. This is often utilised in proofs.

Definition: H is a **proper** subgroup if and only if $H \neq G$ and $H \neq \{e\}$. A subgroup H is a **normal** (invariant) subgroup, if and only if for $\forall g \in G$,

$$ghg^{-1} \in H \text{ for } \forall h \in H$$

A **simple** group G has no proper normal subgroup. A **semi-simple** group G has no abelian normal subgroup.

The **unitary group** $U(n)$ is defined by the set of complex unitary $n \times n$ matrices U, i.e. matrices such that $U^\dagger U = 1$ or $U^{-1} = U^\dagger$. This has the neat property that for $\forall \vec{x}, \vec{y} \in \mathbb{C}^n$ multiplication by a unitary matrix leaves scalar products unchanged:

$$\begin{aligned} \vec{x}' \cdot \vec{y}' &\equiv \vec{x}'^\dagger \vec{y}' = (U\vec{x})^\dagger U\vec{y} \\ &= \vec{x}^\dagger U^\dagger U \vec{y} = \vec{x}^\dagger \vec{y} = \vec{x} \cdot \vec{y} \end{aligned}$$

If we additionally require that $\det(U) = 1$ the matrices form the **special unitary group** $SU(n)$. Let $U_i, U_j \in SU(n)$, then

$$\det(U_i U_j^{-1}) = \det(U_i)\det(U_j^{-1}) = 1.$$

1 is used as the identity matrix in matrix representations.

[3] This is the first of many points where any real mathematician would start to cry loudly and leave the room.

This means that $U_iU_j^{-1} \in SU(N)$. In other words, $SU(n)$ is a **proper subgroup** of $U(n)$. Let $V \in U(n)$ and $U \in SU(n)$, then $VUV^{-1} \in SU(n)$ because:

$$\det(VUV^{-1}) = \det(V)\det(U)\det(V^{-1}) = \frac{\det(V)}{\det(V)}\det(U) = 1.$$

In other words, $SU(n)$ is also a **normal subgroup** of $U(n)$.

Definition: A **(left) coset** of a subgroup $H \subset G$ is a set $\{gh : h \in H\}$ where $g \in G$ and a **(right) coset** of a subgroup $H \subset G$ is a set $\{hg : h \in H\}$ where $g \in G$. For normal subgroups H the left and right cosets coincide and form the **coset group** G/H which has the members $\{gh : h \in H\}$ for $\forall g \in G$ and the binary operation $*$ with $gh * g'h' \in \{(g \bullet g')h : h \in H\}$.

Definition: The **direct product** of groups G and H, $G \times H$, is defined as the *ordered pairs* (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet h_j)$. $G \times H$ is then a group and G and H are normal subgroups of $G \times H$.

Definition: The **semi-direct product** $G \rtimes H$, where G is a mapping $G : H \to H$, is defined by the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet g_i(h_j))$. Here H is not a normal subgroup of $G \rtimes H$.

The SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is an example of a direct product. Direct products are ”trivial” structures because there is no ”interaction” between the subgroups. Can we imagine a group $G \supset SU(3)_c \times SU(2)_L \times U(1)_Y$ that can be broken down to the SM group but has a non-trivial unified gauge structure? There is, $SU(5)$ being one example.

1.2 Representations

Definition: A **representation** of a group G on a vector space V is a map $\rho : G \to GL(V)$, where $GL(V)$ is the **general linear group** on V, i.e. invertible matrices of the field of V, such that for $\forall g_i, g_i \in G$, $\rho(g_i g_j) = \rho(g_i)\rho(g_j)$ (homeomorphism).

For $U(1)$ the transformation $e^{i\chi\alpha}$ is the **fundamental** or **defining representation** which can be used on wavefunctions $\psi(x)$—these form a one dimensional vector space over the complex numbers. For $SU(2)$ the transformation $e^{i\alpha_i\sigma_i}$, with σ being the Pauli matrices, is the **fundamental representation**, which can be applied to *e.g.* weak doublets $\psi = (\nu_l, l)$.[4]

[4]This is a bit daft, since both $U(1)$ and $SU(2)$ are defined in terms of matrices. However, we will also have use for other representations, e.g. the **adjoint representation**, which is not the fundamental or defining representation.

Definition: Two representations ρ and ρ' of G on V and V' are **equivalent** if and only if $\exists A : V \to V'$, that is one-to-one, such that for $\forall g \in G$, $A\rho(g)A^{-1} = \rho'(g)$.

Definition: An **irreducible representation** ρ is a representation where there is *no* proper subspace $W \subset V$ that is closed under the group, i.e. there is no $W \subset V$ such that for $\forall w \in W$, $\forall g \in G$ we have $\rho(g)w \in W$.[a]

[a] In other words, we can not split the matrix representation of G in two parts that do not "mix".

Let $\rho(g)$ for $g \in G$ act on a vector space V as a matrix. If $\rho(g)$ can be decomposed into $\rho_1(g)$ and $\rho_2(g)$ such that

$$\rho(g)v = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix} v$$

for $\forall v \in V$, then ρ is **reducible**.

Definition: $T(R)$ is the **Dynkin index** of the representation R in terms of matrices T_a, given by $\mathrm{Tr}[T_a, T_b] = T(R)\delta_{ab}$. $C(R)$ is the **Casimir invariant** given by $C(R)\delta_{ij} = (T^a T^a)_{ij}$

1.3 Lie groups

We begin by defining what we mean by Lie groups

Definition: A **Lie group** G is a finite-dimensional, n, **smooth manifold** C^∞, i.e. for $\forall g \in G$, g can locally be mapped onto (parametrised by) $\mathbb{R}^n$ or $\mathbb{C}^n$, and group multiplication and inversion are smooth functions, meaning that given $g(\vec{a}), g'(\vec{a}) \in G$, $g(\vec{a}') \bullet g'(\vec{a}') = g''(\vec{b})$ where $\vec{b}(\vec{a}, \vec{a}')$ is analytic, and $g^{-1}(\vec{a}) = g'(\vec{a}')$ where $\vec{a}'(\vec{a})$ is analytic.

In terms of a Lie group G acting on a vector space V, $\dim(V) = m$ (or more generally an m-dimensional manifold), this means we can write the map $G \times V \to V$ for $\vec{x} \in V$ as $x_i \to x_i' = f_i(x_i, a_j)$ where f_i is analytic in x_i and a_j. Additionally f_i should have an inverse.

The translation group $T(1)$ with $g(a) = e^{a\frac{d}{dx}}$ is a Lie group since $g(a) \cdot g(a') = g(a + a')$ and $a + a'$ is analytic. Here we can write $f(x, a) = x + a$. $SU(n)$ are Lie groups as they have a fundamental representation $e^{i\vec{\alpha}\vec{\lambda}}$ where λ is a set of $n \times n$-matrices, and $f_i(\vec{x}, \vec{\alpha}) = [e^{i\vec{\alpha}\vec{\lambda}}\vec{x}]_i$.

By the analyticity we can always construct the parametrization so that $g(0) = e$ or $x_i =$

$f_i(x_i, 0)$. By an infinitesimal transformation da_i we then get the following Taylor expansion[5]

$$\begin{aligned} x_i' &= x_i + dx_i = f_i(x_i, da_i) \\ &= f_i(x_i, 0) + \frac{\partial f_i}{\partial a_j} da_j + \ldots \\ &= x_i + \frac{\partial f_i}{\partial a_j} da_j \end{aligned}$$

This is the transformation by the member of the group that in the parameterisation sits da_j from the identity. If we now let F be a function from the vector space V to either the real $\mathbb{R}$ or complex numbers $\mathbb{C}$, then the group transformation defined by da_i changes F by

$$\begin{aligned} dF &= \frac{\partial F}{\partial x_i} dx_i \\ &= \frac{\partial F}{\partial x_i} \frac{\partial f_i}{\partial a_j} da_j \\ &\equiv da_j X_j F \end{aligned}$$

where the operators defined by

$$X_j \equiv \frac{\partial f_i}{\partial a_j} \frac{\partial}{\partial x_i}$$

are called the n **generators** of the Lie group. It is these generators X that define the action of the Lie group in a given representation as the a's are mere parameters.

As an example of the above we can now go in the opposite direction and look at the two-parameter transformation *defined* by

$$x' = f(x) = a_1 x + a_2,$$

which gives

$$X_1 = \frac{\partial f}{\partial a_1} \frac{\partial}{\partial x} = x \frac{\partial}{\partial x},$$

which is the generator for **dilation** (scale change), and

$$X_2 = \frac{\partial}{\partial x},$$

which is the generator for $T(1)$. Note that $[X_1, X_2] = -X_2$.

Exercise: Find the generators of $SU(2)$ and their commutation relationships. Hint: One answer uses the Pauli matrices, but try to derive this from an infinitesimal parametrization.

Next we lists three central results on Lie groups derived by Sophus Lie [6]:

[5] The fact that f_i is analytic means that this Taylor expansion must converge in some radius around $f_i(x_i, 0)$.

Theorem: (Lie's theorems)

i) For a Lie group $\frac{\partial f_i}{\partial a_j}$ is analytic.

ii) The generators X_i satisfy $[X_i, X_j] = C_{ij}^k X_k$, where C_{ij}^k are **structure constants**.

iii) $C_{ij}^k = -C_{ji}^k$ and $C_{ij}^k C_{kl}^m + C_{jl}^k C_{ki}^m + C_{li}^k C_{kj}^m = 0$.[a]

[a] The second identity follows from the Jacobi identity $[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0$

Exercise: What are the structure constants of SU(2)?

1.4 Lie algebras

Definition: An **algebra** A on a field (say $\mathbb{R}$ or $\mathbb{C}$) is a linear vector space with a binary operation $\circ : A \times A \to A$.

The vector space $\mathbb{R}^3$ together with the cross-product constitutes an algebra.

Definition: A **Lie algebra** L is an algebra where the binary operator $[\,,\,]$, called Lie bracket, has the properties that for $x, y, z \in L$ and $a, b \in \mathbb{R}$ (or $\mathbb{C}$):

i) (associativity)

$$[ax + by, z] = a[x, z] + b[y, z]$$

$$[z, ax + by] = a[z, x] + b[z, y]$$

ii) (anti-commutation)

$$[x, y] = -[y, x]$$

iii) (Jacobi identity)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

We usually restrict ourselves to algebras of linear operators with $[x, y] = xy - yx$, where property iii) is automatic. From Lie's theorems the generators of an n-dimensional Lie group form an n-dimensional Lie algebra.

We mentioned the fundamental representation of a matrix based group earlier. These representations have the lowest possible dimension. Another important representation is the **adjoint**. This consists of the matrices:

$$(M_i)_j^k = -C_{ij}^k$$

where C_{ij}^k are the structure constants. From the Jacobi identity we have $[M_i, M_j] = C_{ij}^k M_k$, meaning that the adjoint representation fulfills the same algebra as the fundamental (generators). Note that the dimension of the fundamental representation n for $SO(n)$ and $SU(n)$ is always smaller than the adjoint, which is equal to the degrees of freedom, $\frac{1}{2}n(n-1)$ and $n^2 - 1$ respectively.

Exercise: Find the dimensions of the fundamental and adjoint representations of $SU(n)$.

Exercise: Find the fundamental representation for $SO(3)$ and the adjoint representation for $SU(2)$. What does this say about the groups and their algebras?

Chapter 2

The Poincaré algebra and its extensions

We now take a look at the groups behind Special Relativity (SR), the Lorentz and Poincaré groups, and look for ways to extend them to internal symmetries, *i.e.* gauge groups.

2.1 The Lorentz Group

A point in the Minkowski space-time manifold $\mathbb{M}_4$ is given by $x^\mu = (t, x, y, z)$ and Einstein's requirement was that physics should be invariant under the Lorentz group.

Definition: The **Lorentz group** L is the group of linear transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ such that $x^2 = x_\mu x^\mu = x'_\mu x'^\mu$ is invariant. The **proper orthochronous Lorentz group** $L_+^\uparrow$ is a subgroup of L where $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 1$. [a]

[a] This guarantees that time moves forward, and makes space and time reflections impossible, with the group describing only boosts and rotations.

From the discussion in the previous section one can show that any $\Lambda \in L_+^\uparrow$ can be written as

$$\Lambda^\mu{}_\nu = \left[\exp\left(-\frac{i}{2}\omega^{\rho\sigma} M_{\rho\sigma}\right)\right]^\mu{}_\nu, \tag{2.1}$$

where $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ are the parameters of the transformation and $M_{\rho\sigma}$ are the generators of L, and the basis of the Lie algebra for L, and are given by:

$$M = \begin{bmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{bmatrix},$$

where K_i and J_i are generators of boost and rotation respectively. These fulfil the following algebra:[1]

$$\begin{aligned}[J_i, J_j] &= -i\epsilon_{ijk}J_k, && (2.2)\\ [J_i, K_j] &= i\epsilon_{ijk}K_k, && (2.3)\\ [K_i, K_j] &= -i\epsilon_{ijk}J_k. && (2.4)\end{aligned}$$

The generators M of L obey the commutation relation:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}). \quad (2.5)$$

2.2 The Poincaré group

We extend L by translation to get the Poincaré group, where translation : $x^\mu \to x'^\mu = x^\mu + a^\mu$. This leaves lengths $(x-y)^2$ invariant in $\mathbb{M}_4$.

Definition: The **Poincaré group** P is the group of all transformations of the form

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu.$$

We can also construct the **restricted Poincaré group** $P_+^\uparrow$, by restricting the matrices Λ in the same way as in $L_+^\uparrow$.

We see that the composition of two elements in the group is:

$$(\Lambda_1, a_1) \bullet (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1).$$

This tells us that the Poincaré group is **not** a direct product of the Lorentz group and the translation group, but a **semi-direct product** of L and the translation group $T(1,3)$, $P = L \rtimes T(1,3)$. The translation generators P_μ have a trivial commutation relationship:[2]

$$[P_\mu, P_\nu] = 0 \quad (2.6)$$

One can show that:[3]

$$[M_{\mu\nu}, P_\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) \quad (2.7)$$

Equations (2.5)–(2.7) form the **Poincaré algebra**, a Lie algebra.

2.3 The Casimir operators of the Poincaré group

Definition: The **Casimir operators** of a Lie algebra are the operators that commute with all elements of the algebra [a]

[a] Technically we say they are members of the centre of the universal enveloping algebra of the Lie algebra. Whatever that means.

[1] Notice that (2.2) and (2.4) are the $SU(2)$ algebra.

[2] This means that the translation group in Minkowski space is abelian. This is obvious, since $x^\mu + y^\mu = y^\mu + x^\mu$. One can show that the differential representation is the expected $P_\mu = -i\partial_\mu$.

[3] For a rigorous derivation of this see Chapter 1.2 of [8]

A central theorem in representation theory for groups and algebras is **Schur's lemma**:

> **Theorem**: (Schur's Lemma)
> In any irreducible representation of a Lie algebra, the Casimir operators are proportional to the identity.

This has the wonderful consequence that the constants of proportionality can be used to classify the (irreducible) representations of the Lie algebra (and group). Let us take a concrete example to illustrate: $P^2 = P_\mu P^\mu$ is a Casimir operator of the Poincaré algebra because the following holds:[4]

$$\left[P_\mu, P^2\right] = 0, \tag{2.11}$$
$$\left[M_{\mu\nu}, P^2\right] = 0. \tag{2.12}$$

This allows us to label the irreducible representation of the Poincaré group with a quantum number m^2, writing a corresponding state as $|m\rangle$, such that:[5]

$$P^2|m\rangle = m^2|m\rangle.$$

The number of invariant Casimir operators is the **rank** of the algebra, *e.g.* rank $SU(n) = n-1$. $P_+^\uparrow$ has rank 2, and thus two Casimir operators.[6]

> **Definition**: The **Pauli-Ljubanski polarisation vector** is given by:
> $$W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}. \tag{2.13}$$

Then $W^2 = W_\mu W^\mu$ is a Casimir operator of $P_+^\uparrow$, *i.e.*:[7]

$$\left[M_{\mu\nu}, W^2\right] = 0 \tag{2.14}$$
$$\left[P_\mu, W^2\right] = 0 \tag{2.15}$$

[4] The first relation follows trivially from the commutation of P_μ with P_ν. To show the second we first use that

$$[M_{\mu\nu}, P_\rho P^\rho] = [M_{\mu\nu}, P_\rho]P^\rho + P_\rho[M_{\mu\nu}, P^\rho] \tag{2.8}$$

and Eq. (2.7) to get:

$$[M_{\mu\nu}, P_\rho P^\rho] = -i([g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu]P^\rho + P_\rho[g_\mu{}^\rho P_\nu - g_\nu{}^\rho P_\mu] \tag{2.9}$$
$$[M_{\mu\nu}, P_\rho P^\rho] = -2i[P_\mu, P_\nu] = 0 \tag{2.10}$$

[5] This quantum number looks astonishingly like mass and P^2 like the square of the 4-momentum operator. However, we note that in general m^2 is not restricted to be larger than zero.

[6] To demonstrate this is rather involved, but note that ($\cong$ meaning homomorfic, that is structure preserving, $=$ meaning isomorphic) $L^\uparrow \cong SL(2,\mathbb{C})$ and $SL(2,\mathbb{C}) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and $SL(2,\mathbb{R}) \cong SU(2)$, which has rank 1.

[7] This is demonstrated in detail in *e.g.* Chapter 1.2 of [8]

We can write this operator as:

$$W^2 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M^{\rho\sigma}M_{\nu\sigma}P_\rho P^\nu.$$

Again, because W^2 is a Casimir operator, we can label all states in an irreducible representation (read particles) with quantum numbers m, s, such that:

$$W^2|m,s\rangle = -m^2 s(s+1)|m,s\rangle$$

The m^2 appears because there are two P_μ operators in each term. However, what is the significance of the s, and why do we choose to write the quantum number in that (familiar?) way? One can easily show using ladder operators that $s = 0, \frac{1}{2}, 1, \ldots$, *i.e.* can only take integer and half integer values. In the rest frame (RF) of the particle we have:[8]

$$P_\mu = (m, \vec{0})$$

Using that $WP = 0$ this gives us $W_0 = 0$ in the RF, and furthermore:

$$W_i = \frac{1}{2}\epsilon_{i0jk} m M^{jk} = mS_i,$$

where $S_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ is the **spin operator**. This gives $W^2 = -\vec{W}^2 = -m^2\vec{S}^2$, meaning that s is indeed the spin quantum number.[9]

The conclusion of this subsection is that anything transforming under the Poincaré group, meaning the objects considered by SR, can be classified by two quantum numbers: mass and spin.

2.4 The no-go theorem and graded Lie algebras

Since we now know the Poincaré group and its representations well, we can ask: Can the external space-time symmetries be extended, perhaps also to include the internal gauge symmetries? Unfortunately no. In 1967 Coleman and Mandula [2] showed that any extension of the Pointcaré group to include gauge symmetries is isomorphic to $G_{SM} \times P_+^\uparrow$, *i.e.* the generators B_i of standard model gauge groups all have

$$[P_\mu, B_i] = [M_{\mu\nu}, B_i] = 0.$$

Not to be defeated by a simple mathematical proof this was countered by Haag, Łopuszański and Sohnius (HLS) in 1975 in [5] where they introduced the concept of graded Lie algebras to get around the no-go theorem.

[8] This does not loose generality since physics should be independent of frame.

[9] Observe that this discussion is problematic for massless particles. However, it is possible to find a similar relation for massless particles, when we chose a frame where the velocity of the particle is mono-directional.

Definition: A ($\mathbb{Z}_2$) **graded Lie algebra** or **superalgebra** is a vector space L that is a direct sum of two vector spaces L_0 and L_1, $L = L_0 \oplus L_1$ with a binary operation $\bullet : L \times L \to L$ such that for $\forall x_i \in L_i$

i) $x_i \bullet x_j \in L_{i+j \mod 2}$ (grading) [a]

ii) $x_i \bullet x_j = -(-1)^{ij} x_j \bullet x_i$ (supersymmetrization)

iii) $x_i \bullet (x_j \bullet x_k)(-1)^{ik} + x_j \bullet (x_k \bullet x_i)(-1)^{ji} + x_k \bullet (x_i \bullet x_j)(-1)^{kj} = 0$ (generalised Jacobi identity)

This can be easily generalised to $\mathbb{Z}_n$, but we will not discuss such extensions further.

[a] This means that $x_0 \bullet x_0 \in L_0$,$x_1 \bullet x_1 \in L_0$ and $x_0 \bullet x_1 \in L_1$.

We can start, as HLS, with a Lie algebra ($L_0 = P_+^\uparrow$) and add a new vector space L_1 spanned by four operators, the Majorana spinor charges Q_a. It can be shown that the superalgebra requirements are fulfilled by:

$$[Q_a, P_\mu] = 0 \tag{2.16}$$

$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu} Q)_a \tag{2.17}$$

$$\{Q_a, \overline{Q}_b\} = 2\not{P}_{ab} \tag{2.18}$$

where $\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ and as usual $\overline{Q}_a = (Q^\dagger \gamma_0)_a$.[10]

Unfortunately, the internal gauge groups are nowhere to be seen. They can appear if we extend the algebra with Q_a^α, where $\alpha = 1, \ldots, N$, which gives gives rise to so-called $N > 1$ supersymmetries. This introduces extra particles and does not seem to be realised in nature due to an extensive number of extra particles.[11] This extension, including $N > 1$, can be proven, under some reasonable assumptions, to be the **largest possible** extension of SR.

[10] Alternatively, (2.18) can be written as $\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu$.

[11] Note that $N > 8$ would include particles with spin greater than 2.

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