

Chapter 1

The algebraic origin of SUSY

The goal of these lectures is to introduce the basics of low-energy models of supersymmetry (SUSY) using the Minimal Supersymmetric Standard Model (MSSM) as a main example. Rather than starting with the problems of the SM, we will focus on the algebraic origin of SUSY in the sense of an extension of the symmetries of Einstein's Special Relativity (SR), which was the original motivation for SUSY.

1.1 What is a group?

Definition: The set $G = \{g_i\}$ and operation \bullet form a **group** if and only if for $\forall g_i \in G$

- i) $g_i \bullet g_j \in G$ (closure)
- ii) $(g_i \bullet g_j) \bullet g_k = g_i \bullet (g_j \bullet g_k)$ (associativity)
- iii) $\exists e \in G$ such that $g_i \bullet e = e \bullet g_i = g_i$ (identity element)
- iv) $\exists g_i^{-1} \in G$ such that $g_i \bullet g_i^{-1} = g_i^{-1} \bullet g_i = e$ (inverse)

A simple example of a group is $G = \mathbb{Z}$ with usual addition as the operation, $e = 0$ and $g^{-1} = -g$. Alternatively we can restrict the group to \mathbb{Z}_n , where the operation is addition with modulo n . In this group, $g_i^{-1} = n - g_i$ and the unit element is $e = 0$. Note that \mathbb{Z} is an *infinite* group, while \mathbb{Z}_n is finite, with *order* n (meaning n members). Both are *abelian* groups, meaning that $g_i \bullet g_j = g_j \bullet g_i$.

All of this is "only" mathematics. Physicists are often more interested in groups where the elements of G *act* on some elements of a set $s \in S$, $g(s) = s' \in S$.¹ S here can for example be the state of a system, say a wave-function in quantum mechanics. We will return to this in a moment, let us just mention that the operation $g_i \bullet g_j$ acts as $(g_i \bullet g_j)(s) = g_i \bullet (g_j(s))$ and the identity acts as $e(s) = s$.²

¹As a result mathematics courses in group theory are not always so relevant to a physicist.

²We can prove this from iii) in the definition. Note that we use e as the identity in an abstract group, while

A more sophisticated example of a group can be found in a use for the Taylor expansion³

$$\begin{aligned} f(x+a) &= f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \dots \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n}{dx^n} f(x) \\ &= e^{a \frac{d}{dx}} f(x) \end{aligned}$$

The operator $T_a = e^{a \frac{d}{dx}}$ is called the **translation operator** (in this case in one dimension). Together with the operation $T_a \bullet T_b = T_{a+b}$ it forms the **translational group** $T(1)$, where $T_a^{-1} = T_{-a}$. In N dimensions the group $T(N)$ has the elements $T_{\vec{a}} = e^{\vec{a} \cdot \vec{\nabla}}$.

Definition: A subset $H \subset G$, is a **subgroup** if and only if:^a

- i) $h_i \bullet h_j \in H$ for $\forall h_i, h_j \in H$
- ii) $h_i^{-1} \in H$ for $\forall h_i \in H$

^aAn alternative, more compact, way of writing these two requirements is $h_i \bullet h_j^{-1} \in H$ for $\forall h_i, h_j \in G$. This is often utilised in proofs.

Definition: H is a **proper** subgroup if and only if $H \neq G$ and $H \neq \{e\}$. A subgroup H is a **normal** (invariant) subgroup, if and only if for $\forall g \in G$,

$$ghg^{-1} \in H \text{ for } \forall h \in H$$

A **simple** group G has no proper normal subgroup. A **semi-simple** group G has no abelian normal subgroup.

The **unitary group** $U(n)$ is defined by the set of complex unitary $n \times n$ matrices U , i.e. matrices such that $U^\dagger U = 1$ or $U^{-1} = U^\dagger$. This has the neat property that for $\forall \vec{x}, \vec{y} \in \mathbb{C}^n$ multiplication by a unitary matrix leaves scalar products unchanged:

$$\begin{aligned} \vec{x}' \cdot \vec{y}' &\equiv \vec{x}'^\dagger \vec{y}' = (U\vec{x})^\dagger U\vec{y} \\ &= \vec{x}^\dagger U^\dagger U \vec{y} = \vec{x}^\dagger \vec{y} = \vec{x} \cdot \vec{y} \end{aligned}$$

If we additionally require that $\det(U) = 1$ the matrices form the **special unitary group** $SU(n)$. Let $U_i, U_j \in SU(n)$, then

$$\det(U_i U_j^{-1}) = \det(U_i) \det(U_j^{-1}) = 1.$$

1 is used as the identity matrix in matrix representations.

³This is the first of many points where any real mathematician would start to cry loudly and leave the room.

This means that $U_i U_j^{-1} \in SU(N)$. In other words, $SU(n)$ is a **proper subgroup** of $U(n)$. Let $V \in U(n)$ and $U \in SU(n)$, then $VUV^{-1} \in SU(n)$ because:

$$\det(VUV^{-1}) = \det(V) \det(U) \det(V^{-1}) = \frac{\det(V)}{\det(V)} \det(U) = 1.$$

In other words, $SU(n)$ is also a **normal subgroup** of $U(n)$.

Definition: A **(left) coset** of a subgroup $H \subset G$ is a set $\{gh : h \in H\}$ where $g \in G$ and a **(right) coset** of a subgroup $H \subset G$ is a set $\{hg : h \in H\}$ where $g \in G$. For normal subgroups H the left and right cosets coincide and form the **coset group** G/H which has the members $\{gh : h \in H\}$ for $\forall g \in G$ and the binary operation $*$ with $gh * g'h' \in \{(g \bullet g')h : h \in H\}$.

Definition: The **direct product** of groups G and H , $G \times H$, is defined as the *ordered pairs* (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet h_j)$. $G \times H$ is then a group and G and H are normal subgroups of $G \times H$.

Definition: The **semi-direct product** $G \rtimes H$, where G is a mapping $G : H \rightarrow H$, is defined by the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \bullet (g_j, h_j) = (g_i \bullet g_j, h_i \bullet g_i(h_j))$. Here H is not a normal subgroup of $G \rtimes H$.

The SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is an example of a direct product. Direct products are "trivial" structures because there is no "interaction" between the subgroups. Can we imagine a group $G \supset SU(3)_c \times SU(2)_L \times U(1)_Y$ that can be broken down to the SM group but has a non-trivial unified gauge structure? There is, $SU(5)$ being one example.

1.2 Representations

Definition: A **representation** of a group G on a vector space V is a map $\rho : G \rightarrow GL(V)$, where $GL(V)$ is the **general linear group** on V , i.e. invertible matrices of the field of V , such that for $\forall g_i, g_j \in G$, $\rho(g_i g_j) = \rho(g_i) \rho(g_j)$ (homeomorphism).

For $U(1)$ the transformation $e^{i\chi\alpha}$ is the **fundamental or defining representation** which can be used on wavefunctions $\psi(x)$ —these form a one dimensional vector space over the complex numbers. For $SU(2)$ the transformation $e^{i\alpha_i \sigma_i}$, with σ being the Pauli matrices, is the **fundamental representation**, which can be applied to *e.g.* weak doublets $\psi = (\nu_l, l)$.⁴

⁴This is a bit daft, since both $U(1)$ and $SU(2)$ are defined in terms of matrices. However, we will also have use for other representations, *e.g.* the **adjoint representation**, which is not the fundamental or defining representation.

Definition: Two representations ρ and ρ' of G on V and V' are **equivalent** if and only if $\exists A : V \rightarrow V'$, that is one-to-one, such that for $\forall g \in G$, $A\rho(g)A^{-1} = \rho'(g)$.

Definition: An **irreducible representation** ρ is a representation where there is *no* proper subspace $W \subset V$ that is closed under the group, i.e. there is no $W \subset V$ such that for $\forall w \in W$, $\forall g \in G$ we have $\rho(g)w \in W$.^a

^aIn other words, we can not split the matrix representation of G in two parts that do not "mix".

Let $\rho(g)$ for $g \in G$ act on a vector space V as a matrix. If $\rho(g)$ can be decomposed into $\rho_1(g)$ and $\rho_2(g)$ such that

$$\rho(g)v = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix} v$$

for $\forall v \in V$, then ρ is **reducible**.

Definition: $T(R)$ is the **Dynkin index** of the representation R in terms of matrices T_a , given by $\text{Tr}[T_a, T_b] = T(R)\delta_{ab}$. $C(R)$ is the **Casimir invariant** given by $C(R)\delta_{ij} = (T^a T^a)_{ij}$

1.3 Lie groups

We begin by defining what we mean by Lie groups

Definition: A **Lie group** G is a finite-dimensional, n , **smooth manifold** C^∞ , i.e. for $\forall g \in G$, g can locally be mapped onto (parametrised by) \mathbb{R}^n or \mathbb{C}^n , and group multiplication and inversion are smooth functions, meaning that given $g(\vec{a}), g'(\vec{a}) \in G$, $g(\vec{a}') \bullet g'(\vec{a}') = g''(\vec{b})$ where $\vec{b}(\vec{a}, \vec{a}')$ is analytic, and $g^{-1}(\vec{a}) = g'(\vec{a}')$ where $\vec{a}'(\vec{a})$ is analytic.

In terms of a Lie group G acting on a vector space V , $\dim(V) = m$ (or more generally an m -dimensional manifold), this means we can write the map $G \times V \rightarrow V$ for $\vec{x} \in V$ as $x_i \rightarrow x'_i = f_i(x_i, a_j)$ where f_i is analytic in x_i and a_j . Additionally f_i should have an inverse.

The translation group $T(1)$ with $g(a) = e^{a \frac{d}{dx}}$ is a Lie group since $g(a) \cdot g(a') = g(a + a')$ and $a + a'$ is analytic. Here we can write $f(x, a) = x + a$. $SU(n)$ are Lie groups as they have a fundamental representation $e^{i\vec{a}\vec{\lambda}}$ where λ is a set of $n \times n$ -matrices, and $f_i(\vec{x}, \vec{a}) = [e^{i\vec{a}\vec{\lambda}}\vec{x}]_i$.

By the analyticity we can always construct the parametrization so that $g(0) = e$ or $x_i =$

$f_i(x_i, 0)$. By an infinitesimal transformation da_i we then get the following Taylor expansion⁵

$$\begin{aligned} x'_i &= x_i + dx_i = f_i(x_i, da_i) \\ &= f_i(x_i, 0) + \frac{\partial f_i}{\partial a_j} da_j + \dots \\ &= x_i + \frac{\partial f_i}{\partial a_j} da_j \end{aligned}$$

This is the transformation by the member of the group that in the parameterisation sits da_j from the identity. If we now let F be a function from the vector space V to either the real \mathbb{R} or complex numbers \mathbb{C} , then the group transformation defined by da_i changes F by

$$\begin{aligned} dF &= \frac{\partial F}{\partial x_i} dx_i \\ &= \frac{\partial F}{\partial x_i} \frac{\partial f_i}{\partial a_j} da_j \\ &\equiv da_j X_j F \end{aligned}$$

where the operators defined by

$$X_j \equiv \frac{\partial f_i}{\partial a_j} \frac{\partial}{\partial x_i}$$

are called the n **generators** of the Lie group. It is these generators X that define the action of the Lie group in a given representation as the a 's are mere parameters.

As an example of the above we can now go in the opposite direction and look at the two-parameter transformation *defined* by

$$x' = f(x) = a_1 x + a_2,$$

which gives

$$X_1 = \frac{\partial f}{\partial a_1} \frac{\partial}{\partial x} = x \frac{\partial}{\partial x},$$

which is the generator for **dilation** (scale change), and

$$X_2 = \frac{\partial}{\partial x},$$

which is the generator for $T(1)$. Note that $[X_1, X_2] = -X_2$.

Exercise: Find the generators of $SU(2)$ and their commutation relationships. Hint: One answer uses the Pauli matrices, but try to derive this from an infinitesimal parametrization.

Next we lists three central results on Lie groups derived by Sophus Lie [6]:

⁵The fact that f_i is analytic means that this Taylor expansion must converge in some radius around $f_i(x_i, 0)$.

Theorem: (Lie's theorems)

- i) For a Lie group $\frac{\partial f_i}{\partial a_j}$ is analytic.
- ii) The generators X_i satisfy $[X_i, X_j] = C_{ij}^k X_k$, where C_{ij}^k are **structure constants**.
- iii) $C_{ij}^k = -C_{ji}^k$ and $C_{ij}^k C_{kl}^m + C_{jl}^k C_{ki}^m + C_{li}^k C_{kj}^m = 0$.^a

^aThe second identity follows from the Jacobi identity $[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0$

Exercise: What are the structure constants of $SU(2)$?

1.4 Lie algebras

Definition: An **algebra** A on a field (say \mathbb{R} or \mathbb{C}) is a linear vector space with a binary operation $\circ : A \times A \rightarrow A$.

The vector space \mathbb{R}^3 together with the cross-product constitutes an algebra.

Definition: A **Lie algebra** L is an algebra where the binary operator $[\ , \]$, called Lie bracket, has the properties that for $x, y, z \in L$ and $a, b \in \mathbb{R}$ (or \mathbb{C}):

- i) (associativity)

$$[ax + by, z] = a[x, z] + b[y, z]$$

$$[z, ax + by] = a[z, x] + b[z, y]$$

- ii) (anti-commutation)

$$[x, y] = -[y, x]$$

- iii) (Jacobi identity)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

We usually restrict ourselves to algebras of linear operators with $[x, y] = xy - yx$, where property iii) is automatic. From Lie's theorems the generators of an n -dimensional Lie group

form an n -dimensional Lie algebra.

We mentioned the fundamental representation of a matrix based group earlier. These representations have the lowest possible dimension. Another important representation is the **adjoint**. This consists of the matrices:

$$(M_i)_j^k = -C_{ij}^k$$

where C_{ij}^k are the structure constants. From the Jacobi identity we have $[M_i, M_j] = C_{ij}^k M_k$, meaning that the adjoint representation fulfills the same algebra as the fundamental (generators). Note that the dimension of the fundamental representation n for $SO(n)$ and $SU(n)$ is always smaller than the adjoint, which is equal to the degrees of freedom, $\frac{1}{2}n(n-1)$ and $n^2 - 1$ respectively.

Exercise: Find the dimensions of the fundamental and adjoint representations of $SU(n)$.

Exercise: Find the fundamental representation for $SO(3)$ and the adjoint representation for $SU(2)$. What does this say about the groups and their algebras?

Chapter 2

The Poincaré algebra and its extensions

We now take a look at the groups behind Special Relativity (SR), the Lorentz and Poincaré groups, and look for ways to extend them to internal symmetries, *i.e.* gauge groups.

2.1 The Lorentz Group

A point in the Minkowski space-time manifold \mathbb{M}_4 is given by $x^\mu = (t, x, y, z)$ and Einstein's requirement was that physics should be invariant under the Lorentz group.

Definition: The **Lorentz group** L is the group of linear transformations $x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$ such that $x^2 = x_\mu x^\mu = x'_\mu x'^\mu$ is invariant. The **proper orthochronous Lorentz group** L_+^\uparrow is a subgroup of L where $\det \Lambda = 1$ and $\Lambda^0_0 \geq 1$.

^aThis guarantees that time moves forward, and makes space and time reflections impossible, with the group describing only boosts and rotations.

From the discussion in the previous section one can show that any $\Lambda \in L_+^\uparrow$ can be written as

$$\Lambda^\mu_\nu = \left[\exp \left(-\frac{i}{2} \omega^{\rho\sigma} M_{\rho\sigma} \right) \right]^\mu_\nu, \quad (2.1)$$

where $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ are the parameters of the transformation and $M_{\rho\sigma}$ are the generators of L , and the basis of the Lie algebra for L , and are given by:

$$M = \begin{bmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{bmatrix},$$

where K_i and J_i are generators of boost and rotation respectively. These fulfil the following algebra:¹

$$[J_i, J_j] = -i\epsilon_{ijk}J_k, \quad (2.2)$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k, \quad (2.3)$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k. \quad (2.4)$$

The generators M of L obey the commutation relation:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}). \quad (2.5)$$

2.2 The Poincaré group

We extend L by translation to get the Poincaré group, where translation : $x^\mu \rightarrow x'^\mu = x^\mu + a^\mu$. This leaves lengths $(x - y)^2$ invariant in \mathbb{M}_4 .

Definition: The **Poincaré group** P is the group of all transformations of the form

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu.$$

We can also construct the **restricted Poincaré group** P_+^\uparrow , by restricting the matrices Λ in the same way as in L_+^\uparrow .

We see that the composition of two elements in the group is:

$$(\Lambda_1, a_1) \bullet (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1).$$

This tells us that the Poincaré group is **not** a direct product of the Lorentz group and the translation group, but a **semi-direct product** of L and the translation group $T(1, 3)$, $P = L \rtimes T(1, 3)$. The translation generators P_μ have a trivial commutation relationship:²

$$[P_\mu, P_\nu] = 0 \quad (2.6)$$

One can show that:³

$$[M_{\mu\nu}, P_\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) \quad (2.7)$$

Equations (2.5)–(2.7) form the **Poincaré algebra**, a Lie algebra.

2.3 The Casimir operators of the Poincaré group

Definition: The **Casimir operators** of a Lie algebra are the operators that commute with all elements of the algebra ^a

^aTechnically we say they are members of the centre of the universal enveloping algebra of the Lie algebra. Whatever that means.

¹Notice that (2.2) and (2.4) are the $SU(2)$ algebra.

²This means that the translation group in Minkowski space is abelian. This is obvious, since $x^\mu + y^\mu = y^\mu + x^\mu$. One can show that the differential representation is the expected $P_\mu = -i\partial_\mu$.

³For a rigorous derivation of this see Chapter 1.2 of [8]

A central theorem in representation theory for groups and algebras is **Schur's lemma**:

Theorem: (Schur's Lemma)

In any irreducible representation of a Lie algebra, the Casimir operators are proportional to the identity.

This has the wonderful consequence that the constants of proportionality can be used to classify the (irreducible) representations of the Lie algebra (and group). Let us take a concrete example to illustrate: $P^2 = P_\mu P^\mu$ is a Casimir operator of the Poincaré algebra because the following holds:⁴

$$[P_\mu, P^2] = 0, \quad (2.11)$$

$$[M_{\mu\nu}, P^2] = 0. \quad (2.12)$$

This allows us to label the irreducible representation of the Poincaré group with a quantum number m^2 , writing a corresponding state as $|m\rangle$, such that:⁵

$$P^2|m\rangle = m^2|m\rangle.$$

The number of Casimir operators is the **rank** of the algebra, *e.g.* $\text{rank } SU(n) = n - 1$. It turns out that P_+^\dagger has rank 2, and thus two Casimir operators. To demonstrate this is rather involved, and we won't make an attempt here, but note that it can be shown that⁶ $L_+^\dagger \cong SL(2, \mathbb{C})$, furthermore $SL(2, \mathbb{C}) = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ because we can split a complex number into two real numbers, and $SL(2, \mathbb{R}) \cong SU(2)$, which has rank 1. We will return to this relationship between L_+^\dagger and $SL(2, \mathbb{C})$ in Section 2.5, where we use it to reformulate the algebras we work with in supersymmetry.

So, what is the second Casimir of the Poincaré algebra?

Definition: The **Pauli-Ljubanski polarisation vector** is given by:

$$W_\mu \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}. \quad (2.13)$$

⁴The first relation follows trivially from the commutation of P_μ with P_ν . To show the second we first use that

$$[M_{\mu\nu}, P_\rho P^\rho] = [M_{\mu\nu}, P_\rho] P^\rho + P_\rho [M_{\mu\nu}, P^\rho], \quad (2.8)$$

and Eq. (2.7) to get:

$$[M_{\mu\nu}, P_\rho P^\rho] = -i(g_{\mu\rho} P_\nu - g_{\nu\rho} P_\mu) P^\rho - iP_\rho (g_\mu{}^\rho P_\nu - g_\nu{}^\rho P_\mu), \quad (2.9)$$

thus

$$[M_{\mu\nu}, P_\rho P^\rho] = -2i[P_\mu, P_\nu] = 0. \quad (2.10)$$

⁵This quantum number looks astonishingly like mass and P^2 like the square of the 4-momentum operator. However, we note that in general m^2 is not restricted to be larger than zero.

⁶Here \cong means homomorphic, that is structure preserving, while $=$ means isomorphic.

Then $W^2 = W_\mu W^\mu$ is a Casimir operator of P_+^\dagger , *i.e.*:

$$[M_{\mu\nu}, W^2] = 0 \quad (2.14)$$

$$[P_\mu, W^2] = 0 \quad (2.15)$$

To show this we can re-write the operator as:⁷

$$W^2 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M^{\rho\sigma}M_{\nu\sigma}P_\rho P^\nu.$$

From the above it is easy to show that W^2 is indeed a Casimir

Again, because W^2 is a Casimir operator, we can label all states in an irreducible representation (read particles) with quantum numbers m, s , such that:

$$W^2|m, s\rangle = -m^2s(s+1)|m, s\rangle$$

The m^2 appears because there are two P_μ operators in each term. However, what is the significance of the s , and why do we choose to write the quantum number in that (familiar?) way? One can easily show using ladder operators that $s = 0, \frac{1}{2}, 1, \dots$, *i.e.* can only take integer and half integer values. In the rest frame (RF) of the particle we have:⁸

$$P_\mu = (m, \vec{0})$$

Using that $WP = 0$ this gives us $W_0 = 0$ in the RF, and furthermore:

$$W_i = \frac{1}{2}\epsilon_{i0jk}mM^{jk} = mS_i,$$

where $S_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ is the **spin operator**. This gives $W^2 = -\vec{W}^2 = -m^2\vec{S}^2$, meaning that s is indeed the spin quantum number.⁹

The conclusion of this subsection is that anything transforming under the Poincaré group, meaning the objects considered by SR, can be classified by two quantum numbers: mass and spin.

2.4 The no-go theorem and graded Lie algebras

Since we now know the Poincaré group and its representations well, we can ask: Can the external space-time symmetries be extended, perhaps also to include the internal gauge symmetries? Unfortunately no. In 1967 Coleman and Mandula [2] showed that any extension of the Poincaré group to include gauge symmetries is isomorphic to $G_{SM} \times P_+^\dagger$, *i.e.* the generators B_i of standard model gauge groups all have

$$[P_\mu, B_i] = [M_{\mu\nu}, B_i] = 0.$$

Not to be defeated by a simple mathematical proof this was countered by Haag, Łopuszański and Sohnius (HLS) in 1975 in [5] where they introduced the concept of graded Lie algebras

⁷This is non-trivial to demonstrate, see Chapter 1.2 of [8].

⁸This does not lose generality since physics should be independent of frame.

⁹Observe that this discussion is problematic for massless particles. However, it is possible to find a similar relation for massless particles, when we chose a frame where the velocity of the particle is mono-directional.

to get around the no-go theorem.

Definition: A (\mathbb{Z}_2) **graded Lie algebra** or **superalgebra** is a vector space L that is a direct sum of two vector spaces L_0 and L_1 , $L = L_0 \oplus L_1$ with a binary operation $\bullet : L \times L \rightarrow L$ such that for $\forall x_i \in L_i$

- i) $x_i \bullet x_j \in L_{i+j \bmod 2}$ (grading)^a
- ii) $x_i \bullet x_j = -(-1)^{ij} x_j \bullet x_i$ (supersymmetrization)
- iii) $x_i \bullet (x_j \bullet x_k)(-1)^{ik} + x_j \bullet (x_k \bullet x_i)(-1)^{ji} + x_k \bullet (x_i \bullet x_j)(-1)^{kj} = 0$ (generalised Jacobi identity)

This definition can be generalised to \mathbb{Z}_n by a direct sum over n vector spaces L_i , $L = \bigoplus_{i=0}^{n-1} L_i$, such that $x_i \bullet x_j \in L_{i+j \bmod n}$ with the same requirements for supersymmetrization and Jacobi identity as for the \mathbb{Z}_2 graded algebra.

^aThis means that $x_0 \bullet x_0 \in L_0$, $x_1 \bullet x_1 \in L_0$ and $x_0 \bullet x_1 \in L_1$.

We can start, as HLS, with a Lie algebra ($L_0 = P_+^\dagger$) and add a new vector space L_1 spanned by four operators, the Majorana spinor charges Q_a . It can be shown that the superalgebra requirements are fulfilled by:

$$[Q_a, P_\mu] = 0 \quad (2.16)$$

$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu} Q)_a \quad (2.17)$$

$$\{Q_a, \bar{Q}_b\} = 2\delta_{ab} \quad (2.18)$$

where $\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ and as usual $\bar{Q}_a = (Q^\dagger \gamma_0)_a$.¹⁰

Unfortunately, the internal gauge groups are nowhere to be seen. They can appear if we extend the algebra with Q_a^α , where $\alpha = 1, \dots, N$, which gives rise to so-called $N > 1$ supersymmetries. This introduces extra particles and does not seem to be realised in nature due to an extensive number of extra particles.¹¹ This extension, including $N > 1$, can be proven, under some reasonable assumptions, to be the **largest possible** extension of SR.

2.5 Weyl spinors

Previously we claimed that there is a homomorphism between L_+^\dagger and $SL(2, \mathbb{C})$. This homomorphism, with $\Lambda^\mu{}_\nu \in L_+^\dagger$ and $M \in SL(2, \mathbb{C})$, can be explicitly given by:¹²

$$\Lambda^\mu{}_\nu(M) = \frac{1}{2} \text{Tr}[\bar{\sigma}^\mu M \sigma_\nu M^\dagger], \quad (2.19)$$

$$M(\Lambda^\mu{}_\nu) = \pm \frac{1}{\sqrt{\det(\Lambda^\mu{}_\nu \sigma_\mu \bar{\sigma}^\nu)}} \Lambda^\mu{}_\nu \sigma_\mu \bar{\sigma}^\nu, \quad (2.20)$$

where $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and $\sigma^\mu = (1, \vec{\sigma})$.

¹⁰Alternatively, (2.18) can be written as $\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu$.

¹¹Note that $N > 8$ would include particles with spin greater than 2.

¹²The sign in Eq. (2.20) is the reason that this is a homomorphism, instead of an isomorphism. Each element in $SL(2, \mathbb{C})$ can be assigned to two in L_+^\dagger .

Since we have this homomorphism we can look at the representations of $SL(2, \mathbb{C})$ instead of the Poincaré group (with its usual Dirac spinors) when we describe particles, but what are those representations? It turns out that there exist two inequivalent fundamental representations of $SL(2, \mathbb{C})$:

- i) The self-representation $\rho(M) = M$ working on an element ψ of a representation space F :

$$\psi'_A = M_A{}^B \psi_B \quad A, B = 1, 2$$

- ii) The complex conjugate self-representation $\rho(M) = M^*$ working on $\bar{\psi}$ in a space \dot{F} :¹³

$$\bar{\psi}'_{\dot{A}} = (M^*)_{\dot{A}}{}^{\dot{B}} \bar{\psi}_{\dot{B}} \quad \dot{A}, \dot{B} = 1, 2$$

Definition: ψ and $\bar{\psi}$ are called **left- and right-handed Weyl spinors**.

Indices can be lowered and raised with:

$$\epsilon_{AB} = \epsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\epsilon^{AB} = \epsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The relationship between ψ and $\bar{\psi}$ can be expressed with:¹⁴

$$\bar{\sigma}^{\dot{A}A} (\psi_A)^* = \bar{\psi}^{\dot{A}}$$

Note that from the above:

$$(\psi_A)^\dagger = \bar{\psi}_{\dot{A}}$$

$$(\bar{\psi}_{\dot{A}})^\dagger = \psi_A$$

We define contractions of Weyl spinors as follows:

Definition: $\psi\chi \equiv \psi^A \chi_A$ and $\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{A}} \bar{\chi}^{\dot{A}}$.

These quantities are invariant under $SL(2, \mathbb{C})$. With this in hand we see that

$$\psi^2 \equiv \psi\psi = \psi^A \psi_A = \epsilon^{AB} \psi_B \psi_A = \epsilon^{12} \psi_2 \psi_1 + \epsilon^{21} \psi_1 \psi_2 = \psi_2 \psi_1 - \psi_1 \psi_2.$$

This quantity is zero if the Weyl spinors commute. In order to avoid this we make the following assumption which is consistent with how we treat fermions (and Dirac spinors):

Postulate: All Weyl spinors anticommute:^a $\{\psi_A, \psi_B\} = \{\bar{\psi}_{\dot{A}}, \bar{\psi}_{\dot{B}}\} = \{\psi_A, \bar{\psi}_{\dot{B}}\} = \{\bar{\psi}_{\dot{A}}, \psi_B\} = 0$.

^aThis means that Weyl spinors are so-called **Grassmann numbers**.

¹³The dot on the indices is just there to help us remember which sum is which and does not carry any additional importance.

¹⁴This is a bit daft, as $\bar{\sigma}^{\dot{A}A} = \delta_{\dot{A}A}$, and we will in the following omit the matrix and write $(\psi_A)^* = \bar{\psi}^{\dot{A}}$.

This means that

$$\psi^2 \equiv \psi\psi = \psi^A\psi_A = -2\psi_1\psi_2.$$

Weyl spinors can be related to Dirac spinors ψ_a as well:¹⁵

$$\psi_a = \begin{pmatrix} \psi_A \\ \bar{\chi}^{\dot{A}} \end{pmatrix}.$$

We see that in order to describe a Dirac spinor we need both handedness of Weyl spinor. For Majorana spinors we have:

$$\psi_a = \begin{pmatrix} \psi_A \\ \bar{\psi}^{\dot{A}} \end{pmatrix}.$$

We can now write the super-Poincaré algebra (superalgebra) in terms of Weyl spinors. With $Q_a = [Q_A, \bar{Q}^{\dot{A}}]$ (a Majorana spinor) we have:

$$\{Q_A, Q_B\} = \{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = 0 \quad (2.21)$$

$$\{Q_A, \bar{Q}_{\dot{B}}\} = 2\sigma_{A\dot{B}}^{\mu} P_{\mu} \quad (2.22)$$

$$[Q_A, P_{\mu}] = [\bar{Q}_{\dot{A}}, P_{\mu}] = 0 \quad (2.23)$$

$$[Q_A, M^{\mu\nu}] = i\sigma_A^{\mu\nu B} Q_B \quad (2.24)$$

where $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu})$.

¹⁵Note that in general $(\psi_A)^* \neq \bar{\chi}^{\dot{A}}$.

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