

Chapter 1

Groups and algebras

The study of symmetries plays a central part in theoretical physics, and the mathematical language we use is that of groups. The action of the group elements on our (quantum mechanical) states describes the transformation of the symmetry operation, while the invariance of the physical properties of the system under that transformation is the symmetry itself. For example, rotations in three-dimensions can be carried out by the application of a 3×3 rotation matrix on the coordinates. These matrices form the group called $SO(3)$. For a sphere, which is invariant under these rotations, $SO(3)$ is then the symmetry group.

Of special interest to us are the Lie groups, which are the groups that represent continuous transformations, such as the rotations. The properties of Lie groups can be further studied by finding their generators which form a (Lie) algebra. The generators *almost* – in a very specific sense of the word almost – describes the whole group, and allows us to reconstruct the group elements by what is called the exponential map.

Here, we will begin by defining groups and looking at some of their most important properties. What is crucial in physics are the *representations* of groups, meaning what the operators of the transformations on the states actually look like. Returning to the rotation example these are 3×3 matrices, but with some restrictions on their elements. After discussing representations we will move on to defining Lie groups, before we end on a discussion of their generators and corresponding algebras.

1.1 Group definition

A group is an abstract mathematical structure that consists of a set of objects (elements), and a multiplication rule acting between pairs of these objects. We define a group as follows.

Definition: The set of elements $G = \{g_i\}$ and operation $\circ$ (sometimes called multiplication) form a **group** if and only if for $\forall g_i \in G$:

- i) $g_i \circ g_j \in G$, (closure)
- ii) $(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)$, (associativity)
- iii) $\exists e \in G$ such that $g_i \circ e = e \circ g_i = g_i$, (identity element)
- iv) $\exists g_i^{-1} \in G$ such that $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = e$. (inverse)

Below, where no confusion can occur, we will often drop the multiplication symbol for the group multiplication (and other abstract multiplications), writing $g_i \circ g_j = g_i g_j$.

A straight forward example of a group is $G = \mathbb{Z}$ (the integers), with standard addition as the operation $\circ$. Then $e = 0$ and $g^{-1} = -g$. Alternatively we can restrict the group to $\mathbb{Z}_n$, where the operation is addition modulo n . In this group, $g_i^{-1} = n - g_i$ and the unit element is again $e = 0$.^[1] Here, $\mathbb{Z}$ is an example of an **infinite** group, the set has an infinite number of members, while $\mathbb{Z}_n$ is **finite**, with **order** n , meaning n members. Both are **abelian** groups, meaning that the elements commute: $g_i \circ g_j = g_j \circ g_i$, because the standard addition commutes.

The simplest, non-trivial, of these $\mathbb{Z}_n$ -groups is $\mathbb{Z}_2$ which only has the members $e = 0$ and 1. The “multiplication” operation is completely defined by the three possibilities $0 + 0 = 0$, $0 + 1 = 1$ and $1 + 1 = 0$. Now, compare this to the set $G = \{-1, 1\}$ with the ordinary multiplication operation. Here, all the possible operations are $1 \cdot 1 = 1$, $1 \cdot (-1) = -1$ and $(-1) \cdot (-1) = 1$. This has exactly the same structure as $\mathbb{Z}_2$, only that the identity element is now 1. We say that these two groups are **isomorphic**, because there is a mapping between all the (two) elements $0 \rightarrow 1$ and $1 \rightarrow -1$, and the results of the multiplication operation is the same, and in fact we consider them as the *same group* despite the considerable apparent visual differences.^[2] This notion of isomorphic (identical) groups is very important, and we will return to it in more detail in Sec. 1.3.

A somewhat more sophisticated example of a group can be found in the Taylor expansion of a function f , where

$$\begin{aligned} f(x+a) &= f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \dots \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n}{dx^n} f(x) \\ &= e^{a \frac{d}{dx}} f(x). \end{aligned}$$

The last equality uses the formal definition of the exponential series, but may drive some mathematicians crazy.^[3] The resulting operator $T_a = e^{a \frac{d}{dx}}$ is called the **translation operator**,

¹Note that we here use e for the identity in an abstract group, while we will later use I or 1 as the identity matrix in matrix representations of groups.

²This observation generalises to the set $G = \{e^{2\pi i k/n} | k = 1, \dots, n-1\}$, the **n-th roots of unity**, which, together with the standard multiplication operation, is isomorphic to $\mathbb{Z}_n$.

³We will not discuss this further, but there is a deep question here whether the operator formed by this exponentiation is well defined.

in this case in one dimension, since it shifts the coordinate x of the function f it is operating on by an amount a . Defining the (natural) multiplication operation $T_a \circ T_b = T_{a+b}$ it forms the **translational group** $T(1)$, where we can show that $T_a^{-1} = T_{-a}$.⁴ In n dimensions the group $T(n)$ has the elements $T_{\mathbf{a}} = e^{\mathbf{a} \cdot \nabla}$. Whereas we say that the groups $\mathbb{Z}$ and $\mathbb{Z}_n$ are **discrete groups**, since we can count the number of elements, T_a is a **continuous group** since the parameter a can be any real number.

1.2 Matrix groups

We next define some groups that are very important in physics and to the discussion in these notes. They have in common that they are defined in terms of square matrices.

1.2.1 General and special linear groups

The largest matrix group for a given matrix dimension is the general linear group.

Definition: The **general linear group** $GL(n)$ is defined by the set of invertible $n \times n$ matrices A under matrix multiplication. If we additionally require that $\det(A) = 1$, the matrices form the **special linear group** $SL(n)$.

The existence of the group identity is guaranteed by the identity matrix I being an invertible matrix (with I as the inverse). Since the existence of an inverse is also necessary in the group definition, we can not construct larger matrix groups. The general linear group also give us our first example of a **non-abelian group**, since matrix multiplication does not in general commute. For two matrices A and B , we may have $AB \neq BA$.

We usually take the matrices in matrix groups to be defined over the field of complex numbers $\mathbb{C}$. If we want to specify the field we may use the notation $GL(n, \mathbb{R})$, signifying that the group is defined over the real numbers. Defined over the complex numbers the $GL(n)$ groups have $2n^2$ free parameters since each of the n^2 elements of the matrices can be a complex number, needing two parameters. The $SL(n)$ group has $2n^2 - 2$ free parameters since the requirement on the determinant fixes both the real and imaginary part of the determinant.

1.2.2 Unitary and special unitary groups

We first remind you that the **Hermitian conjugate** or conjugate transpose of a matrix is given by transposing the matrix and taking the complex conjugate of its elements. Here, we will use the symbol $\dagger$ for this operation, so that for a matrix A , $A^\dagger = (A^T)^*$.

We now define the unitary groups.

Definition: The **unitary group** $U(n)$ is defined by the set of complex unitary $n \times n$ matrices U , *i.e.* matrices such that $U^\dagger U = I$ or $U^{-1} = U^\dagger$. If we additionally require that $\det(U) = 1$ the matrices form the **special unitary group** $SU(n)$.

⁴We could instead have defined the operation between two group elements to be ordinary multiplication and used that to show the relationship $T_a \circ T_b = T_{a+b}$. However, it is important to notice that showing this is not entirely trivial because ordinary arithmetic rules for exponentials fail for operators. In this particular case the proof is fairly simple, but this is in general not so.

It can be shown (see exercises) that the $U(n)$ groups have n^2 independent parameters, while the $SU(n)$ groups have $n^2 - 1$.

It is these groups that form the gauge symmetry groups of the Standard Model: $SU(3)$, $SU(2)$ and $U(1)$. The group $U(1)$ makes perfect sense despite the odd matrix dimensions. This is simply the set of all complex numbers of unit length with ordinary multiplication, *i.e.* $U(1) = \{e^{i\alpha} | \alpha \in \mathbb{R}\}$, but notice that $SU(1)$ would be trivial since it contains only 1.

The unitary group has the important property that for $\forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ multiplication by a unitary matrix leaves scalar products unchanged. If $\mathbf{x}' = U\mathbf{x}$ and $\mathbf{y}' = U\mathbf{y}$, then

$$\begin{aligned}\mathbf{x}' \cdot \mathbf{y}' &\equiv \mathbf{x}'^\dagger \mathbf{y}' = (U\mathbf{x})^\dagger U\mathbf{y} \\ &= \mathbf{x}^\dagger U^\dagger U \mathbf{y} = \mathbf{x}^\dagger \mathbf{y} = \mathbf{x} \cdot \mathbf{y}.\end{aligned}$$

Thus, its members do not change the length of the vectors they act on.

Since we would like to let our group representations act on vectors that describe quantum mechanical states, the unitary groups then conserve probability for these states. For example, when acting on a complex number (a complex scalar), such as a wavefunction $\psi(x)$, the elements of $U(1)$ rotate the phase of ψ , however, the magnitude is conserved since $\psi' = e^{i\alpha}\psi$ gives $|\psi'|^2 = \psi^* e^{-i\alpha} e^{i\alpha} \psi = |\psi|^2$.

1.2.3 Orthogonal and special orthogonal groups

If we restrict the unitary matrices to be real, we get the orthogonal groups.

Definition: The **orthogonal group** $O(n)$ is the group of real $n \times n$ orthogonal matrices O , *i.e.* matrices where $O^T O = I$. If we additionally require that $\det(O) = 1$ the matrices form the **special orthogonal group** $SO(n)$.

It follows from the definition of the orthogonal group that the determinant of the members is either 1 or -1 , thus the special orthogonal group is simply one half of the members. For $\mathbf{x} \in \mathbb{R}^n$ the orthogonal group has the same property as the unitary group of leaving the length of vectors invariant.

Matrices in the $O(n)$ and $SO(n)$ groups have $n(n-1)/2$ independent parameters since an $n \times n$ matrix with real entries has n^2 elements, and there are $n(n+1)/2$ equations to be satisfied by the orthogonality condition.⁵

The special orthogonal groups $SO(2)$ and $SO(3)$ are much used because their elements represent rotations in two and three dimensions, respectively, while $SO(n)$ extends this to higher dimensions and represents the symmetries of a sphere in n dimensions. To see this we can start from the fact that rotations, by definition, conserve angles and distances (and orientation). This means that the original set of orthogonal axis – or orthogonal basis vectors if you wish – must transform into another orthogonal set of axis under the rotation. The matrix performing the rotation must then be orthogonal, and thus the collection of rotations must be $O(n)$. If we additionally require that orientation is preserved, this removes the matrices with negative determinant, leaving the $SO(n)$ group.

⁵Only the upper triangular part of $O^T O$ has independent equations since $O^T O$ is a symmetric matrix, $(O^T O)^T = O^T (O^T)^T = O^T O$.

$SO(2)$

Given that $SO(2)$ has only one parameter, we can write a general group member R as parameterised by θ

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

As expected, we can recognise this as the matrix of rotations of an angle θ around a point in the plane, and it represents the (orientation preserving) symmetries of a circle. Some tinkering with this representation will show that $SO(2)$ is in fact an abelian group, despite the matrix definition. From a physical viewpoint this should be expected: the order of rotations in the plane should not matter.

It is interesting to observe that the elements of $SO(2)$ rotate points in the plane, while the elements of $U(1)$ rotate complex numbers, which can be represented by points in the plane. Indeed, a one-to-one correspondence can be found between the members of the two groups so that the groups are isomorphic, $SO(2) \cong U(1)$.

 $SO(3)$

This group has three free parameters. Already at this point writing down the explicit form of a general group member is not very enlightening. There are also a number of different conventions in use, so proper care is advised when using results from the literature. In terms of a general rotation in three dimensions this can either be viewed as rotation angles around three fixed axis, or as the fixing of a rotation axis by two angles, with a third rotation angle around that axis.

One particular explicit form, where the angles correspond to the three **Euler angles** of rotation in three dimensions, α , β and γ , is

$$R(\alpha, \beta, \gamma) = \begin{bmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \\ \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \\ -\sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \end{bmatrix},$$

where $0 \leq \alpha, \gamma < 2\pi$ and $0 \leq \beta \leq \pi$. These rotations do not commute, so the group is non-abelian.

We have seen above that $SU(2)$ has three free parameters, just the same as $SO(3)$. You may at this point guess that $SO(3)$ is isomorphic to $SU(2)$. That would be a very good guess, however, it would also be wrong. We will return to this later, but this is one of those things in mathematics that turn out to be disappointingly only almost true.

1.3 Group properties

We now extend our vocabulary for groups by defining the **subgroup** of a group G .

Definition: A subset $H \subset G$ is a **subgroup** if and only if:^a

- i) $h_i \circ h_j \in H$ for $\forall h_i, h_j \in H$, (closure)
- ii) $h_i^{-1} \in H$ for $\forall h_i \in H$. (inverse)

H is a **proper** subgroup if and only if $H \neq G$ and $H \neq \{e\}$.

^aAn alternative, equivalent, and more compact way of writing these two requirements is the single requirement $h_i \circ h_j^{-1} \in H$ for $\forall h_i, h_j \in H$. This is often utilised in proofs.

We have already seen some examples of subgroups: the $SU(n)$ groups are subgroups of $U(n)$, and the $SO(n)$ groups are subgroups of the $O(n)$ groups. This can easily be shown using the properties of determinants.

There is a very important class of subgroup called the **normal** subgroup. The importance will become clear in a moment.

Definition: A subgroup H is a **normal** (invariant) subgroup, if and only if the **conjugation** of any element by any $g \in G$ is in H .^a meaning

$$ghg^{-1} \in H \text{ for } \forall h \in H.$$

A **simple** group G has no proper normal subgroup. A **semi-simple** group G has no proper abelian normal subgroup.

^aAnother, pretty, but slightly abusive, way of writing the definition of a normal group is to say that $gHg^{-1} = H$. This implies (correctly), that the image of H under the conjugation operation is guaranteed to be the whole of H .

We can for example show that for $n > 1$, $SU(n)$ is a normal subgroup of $U(n)$ (see exercises).

The normal subgroup can be seen as a factor in the original group that can be divided out to form a simpler group that only retains the structure that was not in the normal group. To be more precise we need the concept of **cosets**.

Definition: A **left coset** of a subgroup $H \subset G$ with respect to $g \in G$ is the set of members $\{gh|h \in H\}$, and a **right coset** of the subgroup is the set $\{hg|h \in H\}$. These are sometimes written gH and Hg , respectively.

For normal subgroups H it can be shown that the sets of left and right cosets coincide and form a group. This is called the **coset group** and denoted G/H .⁶ This has as its members all the distinct sets $\{gh|h \in H\}$, that can be generated by a $g \in G$, and the binary operation $*$ with $\{gh|h \in H\} * \{g'h|h \in H\} = \{(g \circ g')h|h \in H\}$. To simplify notation this can be written $gH * g'H = (g \circ g')H$.

Now that we have introduced group division, we also need to introduce products of groups.

⁶Sometimes called the **factor** or **quotient group**. The notation is pronounced “G mod N”, where “mod” is short for modulo.

Definition: The **direct product** of groups G and H , $G \times H$, is defined as the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \circ (g_j, h_j) = (g_i \circ g_j, h_i \circ h_j)$. $G \times H$ is then a group and G and H can be shown to be normal subgroups of $G \times H$.

Because it has at least one important guest star appearance in this text we also need the definition of the **semi-direct product**.

Definition: The **semi-direct product** $G \rtimes H$, where G is a mapping $G : H \rightarrow H$, is defined by the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \circ (g_j, h_j) = (g_i \circ g_j, h_i \circ g_i(h_j))$. Here H is not a normal subgroup of $G \rtimes H$, but G is.

Note how the semi-direct product is not symmetric between the factors.

The famous Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ is an example of a direct product. Direct products are “trivial” structures because there is no “interaction” between the subgroups, the elements of each group act only on elements of the same group. This is not true for semi-direct products.

Given both the definition of a coset group and the direct product of groups, we can see how to both make and remove products of groups. In extension of this it seems reasonable that the group G and the group $(G \times H)/H$ should be the same group. However, to decide such questions we also need a clear notion of when two groups are the same.

Definition: Two groups G and H are **homomorphic** if there exists a map between the elements of the groups $\rho : G \rightarrow H$, such that for $\forall g, g' \in G$, $\rho(g \circ g') = \rho(g) \circ \rho(g')$.

For homomorphic groups we say that the mapping conserves the structure of the group, or in other words, all the rules for the group operation/multiplication. This leads to our notion of group equality, namely **isomorphic** groups:

Definition: Two groups G and H are **isomorphic**, written $G \cong H$, if they are homomorphic and the relevant mapping is one-to-one.

The one-to-one mapping ensures that there is a one-to-one correspondence between the elements of the group, so that isomorphic groups effectively contain the both the same members and have the same multiplication operation.

1.4 Representations

The previous section was dominantly a mathematical approach to groups. Physicists are usually more interested in groups G where the elements of G *act* to transform some elements of a set $s \in S$, $g(s) = s' \in S$ ⁷. Here, the members of S can for example be the state of a system, say a wave-function in quantum mechanics or a field in quantum field theory. To be usefull in physics, we would like that the result of the group operation $g_i \circ g_j$ acts as $(g_i \circ g_j)(s) = g_i(g_j(s))$ and the identity acts as $e(s) = s$.

⁷As a result mathematics courses in group theory are not always so relevant to a physicist. This part of group theory is called representation theory.