

Chapter 3

Superspace

In this chapter we will introduce a very handy notation system for considering supersymmetry transformations effected by the Q elements of the superalgebra, or, more correctly, the elements of the super-Poincaré group and their representations. This notation uses a coordinate system called superspace, and allows us to define so-called superfields as a replacement of ordinary field theory fields. This mirrors the Lorentz invariance built into relativistic field theory by using four-vectors. In order to do this we first need to know a little more about the properties of Grassman numbers.

3.1 Superspace calculus

Grassmann numbers θ are numbers that anti-commute with each other but not with ordinary numbers. We will here use four such numbers and in addition we want to place them in Weyl spinors, indexed by A and $\dot{A}$.¹

$$\{\theta^A, \theta^B\} = \{\theta^A, \bar{\theta}^{\dot{B}}\} = \{\bar{\theta}^{\dot{A}}, \theta^B\} = \{\bar{\theta}^{\dot{A}}, \bar{\theta}^{\dot{B}}\} = 0.$$

From this we get the relationships:²

$$\theta_A^2 = \theta_A \theta_A = -\theta_A \theta_A = 0, \quad (3.1)$$

$$\theta^2 \equiv \theta\theta \equiv \theta^A \theta_A = -2\theta_1 \theta_2, \quad (3.2)$$

$$\bar{\theta}^2 \equiv \bar{\theta}\bar{\theta} \equiv \bar{\theta}_{\dot{A}} \bar{\theta}^{\dot{A}} = 2\bar{\theta}^{\dot{1}} \bar{\theta}^{\dot{2}}. \quad (3.3)$$

Notice that if we have a function f of a Grassman number, say θ_A , then the all-order expansion of that function in terms of θ_A , is

$$f(\theta_A) = a_0 + a_1 \theta_A, \quad (3.4)$$

as there are simply no more terms because of (3.1).

¹We can already see how this can be useful: if we consistently use $\theta^A Q_A$ and $\bar{\theta}_{\dot{A}} \bar{Q}^{\dot{A}}$ instead of only Q_A and $\bar{Q}^{\dot{A}}$ in Eqs. (2.26)–(2.25) we can actually rewrite the superalgebra as an ordinary Lie algebra, but with Grassman elements, because of these commutation properties.

²There is no summation implied in the first line. These are of course the same relations we already used for the Weyl spinors.

We now need to define differentiation and integration on these numbers in order to create a calculus for them.³

Definition: We define differentiation by:

$$\partial_A \theta^B \equiv \frac{\partial}{\partial \theta^A} \theta^B \equiv \delta_A^B,$$

with a product rule

$$\begin{aligned} \partial_A (\theta^{B_1} \theta^{B_2} \theta^{B_3} \dots \theta^{B_n}) &\equiv (\partial_A \theta^{B_1}) \theta^{B_2} \theta^{B_3} \dots \theta^{B_n} \\ &\quad - \theta^{B_1} (\partial_A \theta^{B_2}) \theta^{B_3} \dots \theta^{B_n} \\ &\quad + \dots + (-1)^{n-1} \theta^{B_1} \theta^{B_2} \dots (\partial_A \theta^{B_n}). \end{aligned} \quad (3.5)$$

This implies that the differential operator ∂_A is itself a Grassmann number and anti-commutes. We can show that the indices of these operators can be raised and lowered with the ϵ in Eqs. (2.18) and (2.19).

Definition: The integral of a function f of a Grassmann number θ_A is defined as a functional $I[f]$

$$I[f] = \int d\theta_A f(\theta_A),$$

which evaluates to a c -number. The evaluation is defined by the relations

$$\int d\theta_A \equiv 0, \quad \int d\theta_A \theta_A \equiv 1,$$

and similarly for $\bar{\theta}_{\dot{A}}$, and we demand linearity:

$$\int d\theta_A [af(\theta_A) + bg(\theta_A)] \equiv a \int d\theta_A f(\theta_A) + b \int d\theta_A g(\theta_A).$$

In these definitions there is no implied summation over the index A .

This has one surprising property. If we take the integral of (3.4) we get:

$$\int d\theta_A f(\theta_A) = a_1 = \partial^A f(\theta_A),$$

meaning that differentiation and integration has the same effect on Grassman numbers.

To integrate over multiple Grassman numbers we define volume elements as

³These definitions have no infinitesimal interpretations, not is there any notion of definite integrals.

Definition:

$$\begin{aligned} d^2\theta &\equiv -\frac{1}{4}d\theta^A d\theta_A, \\ d^2\bar{\theta} &\equiv -\frac{1}{4}d\bar{\theta}_{\dot{A}} d\bar{\theta}^{\dot{A}}, \\ d^4\theta &\equiv d^2\theta d^2\bar{\theta}. \end{aligned}$$

and we demand that the integral operators anti-commute, just as the differential operators

$$\left\{ \int d\theta_A, \int d\theta_B \right\} = \left\{ \int d\theta_A, \theta_B \right\} = 0.$$

This specific volume element definition is made to normalise the following integrals

$$\begin{aligned} \int d^2\theta \theta\theta &= 1, \\ \int d^2\bar{\theta} \bar{\theta}\bar{\theta} &= 1, \\ \int d^4\theta (\theta\theta)(\bar{\theta}\bar{\theta}) &= 1. \end{aligned}$$

Delta functions of Grassmann variables are given by:

$$\begin{aligned} \delta(\theta_A) &= \theta_A, \\ \delta^2(\theta_A) &= \theta\theta, \\ \delta^2(\bar{\theta}^{\dot{A}}) &= \bar{\theta}\bar{\theta}, \end{aligned}$$

and we can easily show that these functions satisfy, just as the usual definition of delta functions,

$$\int d\theta_A f(\theta_A) \delta(\theta_A) = f(0).$$

3.2 Superspace definition

Superspace⁴ is a coordinate system where supersymmetry transformations are *manifest*, in other words, the action of elements in the super-Poincaré group (SP) are treated like Lorentz-transformations are in Minkowski space.

Definition: Superspace is an eight-dimension manifold that can be constructed from the **coset space** of the super-Poincaré group (SP) and the Lorentz group (L), SP/L , by giving coordinates $z^\pi = (x^\mu, \theta^A, \bar{\theta}_{\dot{A}})$, $\pi = \mu, A, \dot{A}$, where x^μ are the ordinary Minkowski coordinates, and where θ_A and $\bar{\theta}^{\dot{A}}$ are four Grassman (anti-commuting) numbers, being the parameters of the Q -operators in the algebra.

⁴First introduced by Salam & Strathdee [\[6\]](#).

To understand what all of this means we begin by writing a general element of SP , $g \in SP$, using the exponential map defined in Section 1.7:

$$g = \exp[ix^\mu P_\mu + i\theta^A Q_A + i\bar{\theta}_{\dot{A}} \bar{Q}^{\dot{A}} + \frac{i}{2}\omega_{\rho\nu} M^{\rho\nu}],$$

where x^μ , θ^A , $\bar{\theta}_{\dot{A}}$ and $\omega_{\rho\nu}$ constitute the parametrisation of the group, and P_μ , Q_A , $\bar{Q}^{\dot{A}}$ and $M_{\rho\nu}$ are the generators. We can now parametrise SP/L simply by setting $\omega_{\mu\nu} = 0$.⁵ The remaining parameters of SP/L are then the coordinates that span superspace.

As physicists we also want to know the dimensions of our new parameters θ . To do this we first look at Eq. (2.27):

$$\{Q_A, \bar{Q}_{\dot{B}}\} = 2\sigma_{A\dot{B}}^\mu P_\mu,$$

where we know that P_μ has mass dimension $[P_\mu] = M$. This means that $[Q^2] = M$ and $[Q] = M^{1/2}$. In the exponential, all terms must have mass dimension zero to make sense. This means that $[\theta Q] = 0$, and therefore $[\theta] = M^{-1/2}$.

We now want to find the effect of supersymmetry transformations (transformations by the super-Poincaré group) on the superspace coordinates, and we begin by noting that any SP transformation can effectively be written in the following way without the boosts and rotations of the Lorentz group:

$$g_0(a, \alpha) = \exp[ia^\mu P_\mu + i\alpha^A Q_A + i\bar{\alpha}_{\dot{A}} \bar{Q}^{\dot{A}}],$$

because one can show that⁶

$$\exp\left[\frac{i}{2}\omega_{\rho\nu} M^{\rho\nu}\right] g_0(a, \alpha) = g_0(\Lambda a, S(\Lambda)\alpha) \exp\left[\frac{i}{2}\omega_{\rho\nu} M^{\rho\nu}\right], \quad (3.6)$$

i.e. all that a Lorentz boost does is to transform spacetime coordinates by Λ and Weyl spinors by $S(\Lambda)$, which is a spinor representation of Λ . Thus, in more colloquial terms, for the supersymmetry transformation it does not matter which reference frame we are working in, we know how the transformation changes between the frames, given by Λa and $S(\Lambda)\alpha$. In addition, since P_μ commutes with all the Q s, so that this part of the group acts independently, when we speak of the **supersymmetry transformation** we will later usually mean just the transformation

$$\delta_S = \alpha^A Q_A + \bar{\alpha}_{\dot{A}} \bar{Q}^{\dot{A}}. \quad (3.7)$$

We can now find the transformation of superspace coordinates under a supersymmetry transformation, just as we have seen the Minkowski coordinates transform under Lorentz transformations. It might be tempting to look at the effects of an element $g_0(a, \alpha)$ on a function on superspace coordinates, $F(z^\pi) = F(x^\mu, \theta^A, \bar{\theta}^{\dot{A}})$ just as we did for the translation group. However, the powers of the sum of generators in the infinite series will become very unwieldy and messy. Instead, we pull out the coordinate dependence from the function $F(z^\pi) = e^{iz^\pi K_\pi} f(0)$, where $K_\pi = (P_\mu, Q_A, \bar{Q}^{\dot{A}})$, and we look at the transformation $z^\pi \rightarrow z'^\pi$ given by

$$g_0 e^{iz^\pi K_\pi} = e^{iz'^\pi K_\pi}.$$

⁵ SP/L is *not* a coset group as defined previously, because L is not a normal subgroup of SP , but its parametrisation still forms a vector space (the coset space) which we call superspace.

⁶Fortunately we are not going to do this because it is messy, but it can be done using the algebra of the group and the series expansion of the exponential function. Note, however, that the proof rests on the P s and Q s forming a closed set, which we saw in the algebra Eqs. (2.26)–(2.25).

In moving the problem to the exponential, we can use the power of some results on the exponentials of non-commuting quantities, in particular the Campbell-Baker-Hausdorff expansion. We have^[7]

$$\begin{aligned} g_0 e^{iz^\pi K_\pi} &= \exp(i a^\nu P_\nu + i \alpha^B Q_B + i \bar{\alpha}_{\dot{B}} \bar{Q}^{\dot{B}}) \exp(iz^\pi K_\pi) \\ &= \exp(i a^\nu P_\nu + i \alpha^B Q_B + i \bar{\alpha}_{\dot{B}} \bar{Q}^{\dot{B}} + iz^\pi K_\pi \\ &\quad - \frac{1}{2} [i a^\nu P_\nu + i \alpha^B Q_B + i \bar{\alpha}_{\dot{B}} \bar{Q}^{\dot{B}}, iz^\pi K_\pi] + \dots) \end{aligned}$$

Now we take a closer look at the commutator^[8]

$$\begin{aligned} [,] &= -[\alpha^B Q_B, \bar{\theta}_{\dot{A}} \bar{Q}^{\dot{A}}] - [\bar{\alpha}_{\dot{B}} \bar{Q}^{\dot{B}}, \theta^A Q_A] \\ &= \alpha^B \bar{\theta}_{\dot{A}} \epsilon^{\dot{A}\dot{C}} \{Q_B, \bar{Q}_{\dot{C}}\} + \bar{\alpha}_{\dot{B}} \theta^A \epsilon^{\dot{B}\dot{C}} \{\bar{Q}_{\dot{C}}, Q_A\} \\ &= 2\alpha^B \bar{\theta}_{\dot{A}} \epsilon^{\dot{A}\dot{C}} \sigma_{B\dot{C}}^\mu P_\mu + 2\bar{\alpha}_{\dot{B}} \theta^A \epsilon^{\dot{B}\dot{C}} \sigma_{A\dot{C}}^\mu P_\mu \\ &= 2(\alpha^B \bar{\theta}^{\dot{C}} \sigma_{B\dot{C}}^\mu + \bar{\alpha}^{\dot{C}} \theta^A \sigma_{A\dot{C}}^\mu) P_\mu \end{aligned}$$

We can relabel $B = A$ and $\dot{C} = \dot{A}$ which leads to

$$-\frac{1}{2} [,] = -(\alpha^A \sigma_{A\dot{A}}^\mu \bar{\theta}^{\dot{A}} - \theta^A \sigma_{A\dot{A}}^\mu \bar{\alpha}^{\dot{A}}) P_\mu.$$

The commutator is proportional with P_μ , and will therefore commute with all operators, in particular the higher terms in the Campbell-Baker-Hausdorff expansion, meaning that the series reduces to

$$g_0 e^{iz^\pi K_\pi} = \exp[i(x^\mu + a^\mu + i\alpha^A \sigma_{A\dot{A}}^\mu \bar{\theta}^{\dot{A}} - i\theta^A \sigma_{A\dot{A}}^\mu \bar{\alpha}^{\dot{A}}) P_\mu + i(\theta^A + \alpha^A) Q_A + i(\bar{\theta}_{\dot{A}} + \bar{\alpha}_{\dot{A}}) \bar{Q}^{\dot{A}}].$$

So superspace coordinates transform under supersymmetry transformations as:

$$(x^\mu, \theta^A, \bar{\theta}_{\dot{A}}) \rightarrow (x^\mu + a^\mu - i\alpha^A \sigma_{A\dot{A}}^\mu \bar{\theta}^{\dot{A}} + i\theta^A \sigma_{A\dot{A}}^\mu \bar{\alpha}^{\dot{A}}, \theta^A + \alpha^A, \bar{\theta}_{\dot{A}} + \bar{\alpha}_{\dot{A}}). \quad (3.8)$$

or given more explicitly as a composition function

$$f(x^\mu, \theta^A, \bar{\theta}_{\dot{A}}, a^\mu, \alpha^A, \bar{\alpha}_{\dot{A}}) = (x^\mu + a^\mu - i\alpha^A \sigma_{A\dot{A}}^\mu \bar{\theta}^{\dot{A}} + i\theta^A \sigma_{A\dot{A}}^\mu \bar{\alpha}^{\dot{A}}, \theta^A + \alpha^A, \bar{\theta}_{\dot{A}} + \bar{\alpha}_{\dot{A}}). \quad (3.9)$$

As a crucial by-product we can now write down a differential representation for the supersymmetry generators by applying the standard expression for the generators X_i of a Lie algebra, given the composition functions f_π :

$$X_j = \frac{\partial f_\pi}{\partial a_j} \frac{\partial}{\partial z_\pi},$$

which gives us^[9]

$$P_\mu = -i\partial_\mu, \quad (3.10)$$

$$iQ_A = -i(\sigma^\mu \bar{\theta})_A \partial_\mu + \partial_A, \quad (3.11)$$

$$i\bar{Q}^{\dot{A}} = -i(\bar{\sigma}^\mu \theta)^{\dot{A}} \partial_\mu + \partial^{\dot{A}}. \quad (3.12)$$

⁷Here we use Campbell-Baker-Hausdorff expansion $e^A e^B = e^{A+B-\frac{1}{2}[A,B]+\dots}$ where the terms that follow all contain commutators of the first commutator $[A,B]$ and the operators A and B .

⁸Using that P_μ commutes with all elements in the algebra, as well as $[\theta^A Q_A, \xi^B Q_B] = -\theta^A \xi^B \{Q_A, Q_B\} = 0$, and the same for $\bar{Q}^{\dot{B}}$.

⁹We define the generators X_i as iP_μ , iQ_A and iQ_B respectively. Again, this is in order to end up with hermitian instead of anti-hermitian generators.

3.3 Covariant derivatives

Similar to the properties of covariant derivatives for gauge transformations in gauge theories, it would be nice to have a derivative that is invariant under supersymmetry transformations, *i.e.* commutes with supersymmetry operators. Obviously $P_\mu = -i\partial_\mu$ does this, but more general covariant derivatives can be made.

Definition: The following **covariant derivatives** commute with supersymmetry transformations:

$$D_A \equiv \partial_A + i(\sigma^\mu \bar{\theta})_A \partial_\mu, \quad (3.13)$$

$$\bar{D}_{\dot{A}} \equiv -\partial_{\dot{A}} - i(\theta \sigma^\mu)_{\dot{A}} \partial_\mu. \quad (3.14)$$

These can be shown to satisfy the following relations that are useful in calculations:

$$\{D_A, D_B\} = \{\bar{D}_{\dot{A}}, \bar{D}_{\dot{B}}\} = 0, \quad (3.15)$$

$$\{D_A, \bar{D}_{\dot{B}}\} = -2\sigma^\mu_{A\dot{B}} P_\mu, \quad (3.16)$$

$$D^3 = \bar{D}^3 = 0, \quad (3.17)$$

$$D^A \bar{D}^2 D_A = \bar{D}_{\dot{A}} D^2 \bar{D}^{\dot{A}}. \quad (3.18)$$

Here, D^3 and $\bar{D}^3$ means the application of at least three of these covariant derivatives.

3.4 Superfields

Using the superspace coordinates we can now define functions of these to use in a field theory. Naturally we should call these objects superfields.

Definition: A **superfield** Φ is an operator valued function on superspace $\Phi(x, \theta, \bar{\theta})$.

Notice how we write, just as for ordinary fields depending on four-vector coordinates, the coordinates sans indices.

We can expand any such superfield $\Phi(x, \theta, \bar{\theta})$ as a power series in θ and $\bar{\theta}$. This gives, in general,

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) = & f(x) + \theta^A \phi_A(x) + \bar{\theta}_{\dot{A}} \bar{\chi}^{\dot{A}}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta} n(x) \\ & + \theta\sigma^\mu \bar{\theta} V_\mu(x) + \theta\theta \bar{\theta}_{\dot{A}} \bar{\lambda}^{\dot{A}}(x) + \bar{\theta}\bar{\theta} \theta^A \psi_A(x) + \theta\theta \bar{\theta}\bar{\theta} d(x), \end{aligned} \quad (3.19)$$

where the functions of space-time coordinates x are called the **component fields**. Any superfield without explicit spinor indices, such as the one above, commutes with any other superfield, because all the Grassmann numbers appear in contracted pairs.

The properties of the component fields can be deduced from the requirement that Φ must be an (operator valued) Lorentz scalar or pseudoscalar. This is shown in Table 3.1 along with the corresponding degrees of freedom each field has.

Component field	Type	d.o.f.
$f(x), m(x), n(x)$	Complex (pseudo) scalar	2
$\psi_A(x), \phi_A(x)$	Left-handed Weyl spinor	4
$\bar{\chi}^{\dot{A}}(x), \bar{\lambda}^{\dot{A}}(x)$	Right-handed Weyl spinor	4
$V_\mu(x)$	Lorentz 4-vector (complex)	8
$d(x)$	Complex scalar	2

Table 3.1: Field content of a general superfield.

One can show that under supersymmetry transformations these component fields transform linearly into each other, thus we say that superfields (with the differential form of the supersymmetry generators) are *representations* of the supersymmetry (super-Poincaré) group just as ordinary quantum fields are representations of the Poincaré group, albeit *highly reducible* representations.¹⁰

We can recover the known irreducible representations, see Section 2.8 by imposing some restrictions on the fields. To do this we define the following three types of superfields that we will discuss below:

$$\bar{D}_{\dot{A}}\Phi(x, \theta, \bar{\theta}) = 0 \quad (\text{left-handed scalar superfield}) \quad (3.20)$$

$$D_A\Phi^\dagger(x, \theta, \bar{\theta}) = 0 \quad (\text{right-handed scalar superfield}) \quad (3.21)$$

$$\Phi^\dagger(x, \theta, \bar{\theta}) = \Phi(x, \theta, \bar{\theta}) \quad (\text{vector superfield}) \quad (3.22)$$

Note that it is $\Phi^\dagger$ which is the right handed superfield in Eq. (3.21), not Φ . However, we can show that the hermitian conjugate of a left-handed scalar superfield fulfils the condition for a right-handed scalar superfield. Supersymmetry transformations can also be shown to transform left-handed superfields into left-handed superfields, right-handed superfields into right-handed superfields and vector superfields into vector superfields. This means that they can be considered separate representations.

Products of the same type of superfield is a superfield of the same type since for left-handed scalar superfields Φ_i and Φ_j ,

$$\bar{D}_{\dot{A}}(\Phi_i\Phi_j) = (\bar{D}_{\dot{A}}\Phi_i)\Phi_j + \Phi_i(\bar{D}_{\dot{A}}\Phi_j) = 0,$$

and similarly for a right-handed scalar superfields, and for vector superfields Φ_i and Φ_j ,

$$(\Phi_i\Phi_j)^\dagger = \Phi_j^\dagger\Phi_i^\dagger = \Phi_j\Phi_i = \Phi_i\Phi_j.$$

The product of a left-handed scalar superfield Φ and its hermitian conjugate $\Phi^\dagger$, $V = \Phi\Phi^\dagger$, is a vector superfield since

$$V^\dagger = (\Phi\Phi^\dagger)^\dagger = \Phi\Phi^\dagger = V.$$

The same is true for sums of superfields of the same type. These properties will be important when creating a superfield version of a Lagrangian.

¹⁰Indeed, they are linear representations since a sum of superfields is a superfield, and the differential supersymmetry operators act linearly.

3.4.1 Scalar superfields

What is the connection of the scalar superfields to the $j = 0$ irreducible representation? To see this we use a cute trick:¹¹ Change to the variable $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$. Then the covariant derivatives simplify to

$$D_A = \partial_A + 2i\sigma_{A\dot{A}}^\mu \bar{\theta}^{\dot{A}} \frac{\partial}{\partial y^\mu}, \quad (3.23)$$

$$\bar{D}_{\dot{A}} = -\partial_{\dot{A}}. \quad (3.24)$$

This means that a field fulfilling $\bar{D}_{\dot{A}}\Phi = 0$ in the new set of coordinates must be independent of the $\bar{\theta}$ coordinates. Thus we can write this field as:

$$\Phi(y, \theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \quad (3.25)$$

and looking at the much restricted component field content we get the result in Table 3.2

Component field	Type	d.o.f.
$A(x), F(x)$	Complex scalar	2
$\psi_A(x)$	Left-handed Weyl spinor	4

Table 3.2: Fields contained in a left-handed scalar superfield.

We can undo the coordinate change in (3.25) by inserting for y and expanding in powers of θ and $\bar{\theta}$, giving

$$\Phi(x, \theta, \bar{\theta}) = A(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu A(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square A(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x). \quad (3.26)$$

By using the transformation $y^\mu \equiv x^\mu - i\theta\sigma^\mu\bar{\theta}$ we can show a similar field content for the right-handed scalar superfield. The general form of a right handed scalar superfield is then:

$$\Phi^\dagger(x, \theta, \bar{\theta}) = A^*(x) - i(\theta\sigma^\mu\bar{\theta})\partial_\mu A^*(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square A^*(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\sigma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}\bar{\theta}F^*(x). \quad (3.27)$$

We can now compare the above to the $j = 0$ representation with two scalar states and two fermionic states (d.o.f.). After applying the equations of motions (e.o.m.) the **auxillary** field $F(x)$ can be completely eliminated as it does not have any derivatives. The e.o.m. also eliminates two of the fermion d.o.f. Thus, after the equations of motion, we are left with the same states as in the $j = 0$ representation.

However, the scalar superfields will not correspond directly to particle states for the known SM particles since, as we discussed in Sec. 2.8.1, a Weyl spinor on its own cannot describe a Dirac fermion. When we construct particle representations we will take one left-handed scalar superfield and one *different* right-handed scalar superfield. These will form a Dirac fermion and two scalars (and their anti-particles) after application of the e.o.m.

¹¹Here cute is used in the widest sense.

3.4.2 Vector superfields

If we take the general superfield in (3.19) and compare the Φ and $\Phi^\dagger$ expressions we can see that the following is the restricted component field structure of a vector superfield:

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) = & f(x) + \theta\phi(x) + \bar{\theta}\bar{\phi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}m^*(x) \\ & + \theta\sigma^\mu\bar{\theta}V_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}d(x).\end{aligned}$$

The properties of the component fields are summarised in Table 3.3.

Component field	Type	d.o.f.
$f(x), d(x)$	Real scalar field	1
$\phi_A(x), \lambda_A(x)$	Weyl spinor	4
$m(x)$	Complex scalar field	2
$V_\mu(x)$	Real Lorentz 4-vector	4

Table 3.3: Field content of a general vector superfield.

With the large number of component fields, you may now be a little suspicious that this vector superfield does not correspond to the promised degrees of freedom in the $j = \frac{1}{2}$ representation of the superalgebra. However, gauge-freedom now comes to our rescue.

3.5 Supergauge

We first define what we will mean by an abelian (super)gauge transformation of a superfield.¹² Later we will see how it relates to the ordinary gauge transformations of quantum fields. We begin with the scalar superfields.

Definition: The abelian **supergauge transformation** (local or global) on a left handed scalar superfield Φ_i is defined as:

$$\Phi_i \rightarrow \Phi'_i = e^{-iq_i\Lambda}\Phi_i$$

where q_i is the charge of Φ_i under that gauge group and Λ (global), or $\Lambda(x)$ (local), is the parameter of the gauge transformation.

The definition is of course completely equivalent for right-handed scalar fields. For the definition to make sense the transformed field Φ'_i must be a left-handed scalar superfield itself, thus

$$\bar{D}_{\dot{A}}\Phi'_i = 0,$$

and this requires:

$$\bar{D}_{\dot{A}}\Phi'_i = \bar{D}_{\dot{A}}e^{-iq_i\Lambda}\Phi_i = e^{-iq_i\Lambda}\bar{D}_{\dot{A}}\Phi_i - iq_i(\bar{D}_{\dot{A}}\Lambda)e^{-iq_i\Lambda}\Phi_i = -iq_i(\bar{D}_{\dot{A}}\Lambda)\Phi'_i = 0.$$

Thus we must have $\bar{D}_{\dot{A}}\Lambda = 0$, which means that the parameter Λ is also a left-handed superfield.

¹²We promise that we will get back to the corresponding definition for non-abelian transformations.

Next, we move to the vector superfields.

Definition: Given a vector superfield $V(x, \theta, \bar{\theta})$, we define the **abelian supergauge transformation** as

$$V(x, \theta, \bar{\theta}) \rightarrow V'(x, \theta, \bar{\theta}) = V(x, \theta, \bar{\theta}) + i[\Lambda(x, \theta, \bar{\theta}) - \Lambda^\dagger(x, \theta, \bar{\theta})],$$

where the parameter of the transformation Λ is a scalar superfield.

With the expressions for scalar superfields in (3.26) and (3.27), using $\Phi = i\Lambda$, we can show that under supergauge transformations the vector superfield components transform as:

$$f(x) \rightarrow f'(x) = f(x) + A(x) + A^*(x) \quad (3.28)$$

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \sqrt{2}\psi(x) \quad (3.29)$$

$$m(x) \rightarrow m'(x) = m(x) + F(x) \quad (3.30)$$

$$V_\mu(x) \rightarrow V'_\mu(x) = V_\mu(x) + i\partial_\mu(A(x) - A^*(x)) \quad (3.31)$$

$$\lambda(x) \rightarrow \lambda'(x) = \lambda(x) \quad (3.32)$$

$$d(x) \rightarrow d'(x) = d(x). \quad (3.33)$$

If we look at the transformation this implies for the vector field, this is equivalent to the ordinary abelian gauge transformation for a vector field $V_\mu(x) \rightarrow V'_\mu(x) = V_\mu(x) + \partial_\mu g(x)$, with the gauge parameter given by the scalar component field of Λ , $g(x) = i[A(x) - A^*(x)] = -2\text{Im}(A(x))$. It then immediately follows that the standard field strength for a vector field, $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, is supergauge invariant.

If we demand that our theory is invariant under these gauge transformations we can choose the component fields of Λ in order to eliminate some the remaining reducibility in the representation.

Definition: The **Wess-Zumino (WZ) gauge** is a supergauge transformation of a vector superfield by a scalar superfield with

$$\psi(x) = -\frac{1}{\sqrt{2}}\phi(x), \quad (3.34)$$

$$F(x) = -m(x), \quad (3.35)$$

$$A(x) + A^*(x) = -f(x). \quad (3.36)$$

A vector superfield in the WZ-gauge can then be written:

$$V_{WZ}(x, \theta, \bar{\theta}) = (\theta\sigma^\mu\bar{\theta})[V_\mu(x) + i\partial_\mu(A(x) - A^*(x))] + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}d(x), \quad (3.37)$$

which, considered carefully, contains one real scalar field d.o.f., three gauge field d.o.f.¹³ and four fermion d.o.f., meaning it corresponds to the representation $j = \frac{1}{2}$. Unfortunately,

¹³Hang on, where did that last d.o.f. go from $V(x)$? We have a remaining gauge freedom in the choice of the imaginary component of $A(x)$, which is the ordinary gauge freedom of a $U(1)$ field theory. This can be used to eliminate one d.o.f. from the vector field.

supersymmetry transformations break this gauge, meaning that a vector superfield in the WZ-gauge will no longer be in the WZ-gauge after a supersymmetry transformation.

Notice that the WZ gauge is particularly convenient for calculations because:

$$V_{WZ}^2 = \frac{1}{2} \theta \theta \bar{\theta} \bar{\theta} [V_\mu(x) + i \partial_\mu (A(x) - A^*(x))] [V^\mu(x) + i \partial^\mu (A(x) - A^*(x))]$$

and, since multiplying in any θ or $\bar{\theta}$ into V_{WZ}^2 will then yield zero, we have

$$V_{WZ}^3 = 0,$$

so that

$$e^{V_{WZ}} = 1 + V_{WZ} + \frac{1}{2} V_{WZ}^2.$$

3.6 Exercises

Exercise 3.1

Show that

$$\int d^2\theta \theta \theta = 1.$$

Exercise 3.2

Check that the explicit differential forms of the generators in Eqs. (3.10)–(3.12) fulfil the superalgebra in Eqs. (2.26)–(2.24).

Exercise 3.3

Demonstrate the correctness of the general expression for the left-handed scalar superfield in Eq. (3.26).

Exercise 3.4

Show the vector superfield supergauge transformation properties for the component fields.
Hint: Use the field redefinitions:

$$\lambda(x) \rightarrow \lambda(x) + \frac{i}{2} \sigma^\mu \partial_\mu \bar{\phi}(x),$$

$$d(x) \rightarrow d(x) - \frac{1}{4} \square f(x).$$

Exercise 3.5

Find the expression for V_{WZ}^2 .

