

Chapter 4

Construction of a low-energy supersymmetric Lagrangian

We would now like to construct models, in the form of a field theory Lagrangians, that are invariant under supersymmetry transformations, much in the same way that the Standard Model Lagrangian is invariant under Poincaré transformations. However, just as for ordinary quantum field theory Lagrangians we will need to be able to impose gauge invariance for a choice of gauge group, and we want to limit the models to models that are *renormalisable*, *i.e.* models where any infinities can be cancelled by a finite number of counter-terms. Along the way we will also find a way to deal with the problem of equal masses for the particles in an irreducible representation – in other words, where are all the supersymmetric partners of the Standard Model particles? – taking inspiration from the Standard Model Higgs mechanism using spontaneous symmetry breaking.

4.1 Supersymmetry invariant Lagrangians and actions

As should be well known the relativistic field theory **action**

$$S \equiv \int_R d^4x \mathcal{L}, \quad (4.1)$$

is invariant under supersymmetry transformations if this transforms the Lagrangian by a total derivative term $\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \partial^\mu f(x)$, where $f(x) \rightarrow 0$ on the surface $S(R)$ of the integration region R . The question then becomes: how can we construct a Lagrangian from superfields with this property?

An infinitesimal **supersymmetry transformation** of a function on superspace $F(x, \theta, \bar{\theta})$ can be written as

$$F'(x, \theta, \bar{\theta}) = \exp(i\alpha Q + i\bar{\alpha}\bar{Q})F(x, \theta, \bar{\theta}) \simeq F(x, \theta, \bar{\theta}) + (i\alpha Q + i\bar{\alpha}\bar{Q})F(x, \theta, \bar{\theta}), \quad (4.2)$$

where α^A and $\bar{\alpha}_{\dot{A}}$ are the infinitesimal parameters of the transformation. If these parameters are constant then we say that this is a **global** supersymmetry transformation which is what we are (mostly) going to concern ourselves with in these notes. Replacing $\alpha \rightarrow \alpha(x)$ gives a **local** supersymmetry transformation.¹ Here we have ignored the action of the P_μ operators

¹Allowing for local supersymmetry transformations leads to a gauge theory of supersymmetry called supergravity, which can incorporate gravity in a natural way.

since they commute with all the Q s, so that this part of the group acts independently, and only results in a translation of the Minkowski coordinate. The total change in the function under the supersymmetry transformation is $\delta_s F = F' - F$, where

$$\delta_s = i(\alpha Q + \bar{\alpha} \bar{Q}). \quad (4.3)$$

We can show that the highest order component fields in θ and $\bar{\theta}$ of a superfield always transform as a total derivative under δ_s , *e.g.* for the general superfield the highest order component field $d(x)$ transforms as²

$$\delta_s d(x) = d'(x) - d(x) = \frac{i}{2}(\partial_\mu \psi(x) \sigma^\mu \bar{\alpha} - \partial_\mu \bar{\lambda}(x) \sigma^\mu \alpha), \quad (4.4)$$

and the same is naturally true for the D component-field of a vector superfield. For a scalar superfield it is the F -field which has this property:

$$\delta_s F(x) = -i\sqrt{2}\partial_\mu \psi(x) \sigma^\mu \bar{\alpha}. \quad (4.5)$$

These highest power θ -components can be isolated by using the projection property of integration in Grassmann calculus writing

$$S = \int_R d^4x \int d^4\theta \mathcal{L},$$

where the Lagrangian density $\mathcal{L}$, which is a function of superfields, is guaranteed to give a supersymmetry invariant action. Note that this constitutes a redefinition of what we mean by the Lagrangian $\mathcal{L}$ when working with superfields. We should in particular be careful when counting the dimension of terms. We have, see Section 3.2, $[\theta] = M^{-1/2}$, which, since $\int d\theta\theta = 1$, leads to $[\int d\theta] = M^{1/2}$. We then have $[\int d^4\theta] = M^2$. Since we must have $[\int d^4\theta \mathcal{L}] = M^4$ for the action to be dimensionless, we need $[\mathcal{L}] = M^2$, which is different from the standard field theory Lagrangian with dimension M^4 .

We can now write down the generic form for a supersymmetric Lagrangian consisting of scalar superfields, where the indices indicate the highest power of θ in the term:

$$\mathcal{L} = \mathcal{L}_{\theta\theta\bar{\theta}\bar{\theta}} + \theta\theta\mathcal{L}_{\bar{\theta}\bar{\theta}} + \bar{\theta}\bar{\theta}\mathcal{L}_{\theta\theta}.$$

Here $\mathcal{L}_{\theta\theta}$ ($\mathcal{L}_{\bar{\theta}\bar{\theta}}$) is a function of *only* left-handed (right-handed) scalar superfields where we project out the F -field by multiplying by $\bar{\theta}\bar{\theta}$ ($\theta\theta$) and integrating over all the θ . These terms form what we call the **superpotential**. Meanwhile, the $\mathcal{L}_{\theta\theta\bar{\theta}\bar{\theta}}$ term is a real valued function of the superfields where we project out the d -field, called the **Kähler potential**.

The requirement of renormalisability puts further restrictions on the fields in $\mathcal{L}$. We can at most have three factors of scalar superfields in each term of the superpotential, for details see *e.g.* Wess & Bagger [7]³. Since the action must be real, the (almost) most general supersymmetry Lagrangian that can be written in terms of scalar superfields is:

$$\mathcal{L} = \Phi_i^\dagger \Phi_i + \bar{\theta}\bar{\theta}W[\Phi] + \theta\theta W[\Phi^\dagger].$$

²A word of warning: we really have to be slightly more careful here with the supersymmetry transformation since the superfields are operator valued functions acting on a state-space. A unitary transformation $U|\psi\rangle$ of the state implies a unitary transformation $U^\dagger A U$ of an operator A on that state.

³The argument is similar to ordinary quantum field theory, terms with couplings with negative mass dimension is forbidden.

Here the first term is called the **kinetic term**,^[4] and W is the symbol for the **superpotential**, which by renormalisability is restricted to

$$W[\Phi] = g_i \Phi_i + m_{ij} \Phi_i \Phi_j + \lambda_{ijk} \Phi_i \Phi_j \Phi_k. \quad (4.6)$$

This means that to specify a supersymmetric Lagrangian we only need to specify the superfield content of the model and the form of the superpotential.

Dimension counting, starting from $[\mathcal{L}] = M^2$ and $[\Phi] = 1$, means that the superpotential terms must each be M^3 , for the couplings this gives $[g_i] = M^2$, $[m_{ij}] = M$ and $[\lambda_{ijk}] = 1$. Notice also that m_{ij} and λ_{ijk} must be symmetric since the superfields commute.

4.2 Abelian gauge theories

We will of course require not only a supersymmetry invariant Lagrangian, but also a gauge invariant Lagrangian, and we start with the requirements for an abelian gauge group.

Let us first look at the transformation of the superpotential W under the gauge transformation of the scalar superfields in (3.27):

$$W[\Phi] \rightarrow W[\Phi'] = g_i e^{-iq_i \Lambda} \Phi_i + m_{ij} e^{-i(q_i + q_j) \Lambda} \Phi_i \Phi_j + \lambda_{ijk} e^{-i(q_i + q_j + q_k) \Lambda} \Phi_i \Phi_j \Phi_k.$$

Requiring gauge invariance, $W[\Phi] = W[\Phi']$, we must have:

$$q_i = 0 \text{ or } g_i = 0, \quad (4.7)$$

$$q_i + q_j = 0 \text{ or } m_{ij} = 0, \quad (4.8)$$

$$q_i + q_j + q_k = 0 \text{ or } \lambda_{ijk} = 0. \quad (4.9)$$

This puts great restrictions on the form of the superpotential and the charge assignments of the superfields, as in ordinary gauge theories.

What then about the kinetic term? This transforms as

$$\Phi_i^\dagger \Phi_i \rightarrow \Phi_i^\dagger e^{iq_i \Lambda^\dagger} e^{-iq_i \Lambda} \Phi_i = e^{iq_i (\Lambda^\dagger - \Lambda)} \Phi_i^\dagger \Phi_i.$$

As in ordinary gauge theories we can introduce a **gauge compensating vector (super)field** V with the appropriate gauge transformation to make the kinetic term invariant under supersymmetry transformations. We write the kinetic term instead as $\Phi_i^\dagger e^{q_i V} \Phi_i$, which gives us an invariant term:^[5]

$$\Phi_i^\dagger e^{q_i V} \Phi_i \rightarrow \Phi_i^\dagger e^{iq_i \Lambda^\dagger} e^{q_i (V + i\Lambda - i\Lambda^\dagger)} e^{-iq_i \Lambda} \Phi_i = \Phi_i^\dagger e^{q_i V} \Phi_i.$$

This definition of gauge transformation can be shown to recover the Standard Model **minimal coupling** for the component fields through the covariant derivative

$$D_\mu^i = \partial_\mu - \frac{i}{2} q_i V_\mu,$$

where V_μ is the vector component field of the vector superfield.

⁴The constant in front of this term can always be chosen to be one because we can rescale the whole Lagrangian. It is possible to start from more general Kähler potentials, but this is mostly beyond the scope of these notes, although we will return to modify this term to account for gauge invariance.

⁵In case you were worried: we can use the WZ gauge to show that the new kinetic term $\Phi_i^\dagger e^{q_i V} \Phi_i$ has no term with dimension higher than four, and is thus renormalizable.

4.3 Non-Abelian gauge theories

How do we extend the above to deal with the much more complicated non-abelian gauge theories? Let us take a gauge group G with the Lie algebra of group generators t_a in the fundamental representation

$$[t_a, t_b] = if_{ab}^c t_c, \quad (4.10)$$

where f_{ab}^c are the structure constants. For an element g in the group G we want to write down a unitary⁶ representation $U(g)$ that transforms a scalar superfield Ψ by $\Psi \rightarrow \Psi' = U(g)\Psi$.⁷ With an exponential map we can write this representation as $U(g) = e^{i\lambda^a t_a}$, where λ^a are the parameters, as you may perhaps have expected.

Thus, as in ordinary gauge theories, we simply copy the structure of the abelian gauge transformation, and transform superfields as

$$\Psi \rightarrow \Psi' = e^{-iq\Lambda^a T_a} \Psi,$$

where q is the charge of Ψ under G . At this point we can choose a representation different from the fundamental, reflected in a different choice for the generators which we write as T_a . Since for gauge groups we are almost exclusively interested in groups defined by a matrix representation $U(g)$ will be a matrix with dimension fixed by the dimension chosen for the representation. Again we can easily show that we must require that the Λ^a are left-handed scalar superfields for Ψ to transform to a left-handed scalar superfield.

For the superpotential to be invariant we must now have:

$$g_i U_{ir} = g_r \text{ or } g_i = 0, \quad (4.11)$$

$$m_{ij} U_{ir} U_{js} = m_{rs} \text{ or } m_{ij} = 0, \quad (4.12)$$

$$\lambda_{ijk} U_{ir} U_{js} U_{kt} = \lambda_{rst} \text{ or } \lambda_{ijk} = 0, \quad (4.13)$$

where the indices on U are its matrix indices.

We also want a similar construction for the kinetic terms as for abelian gauge theories, $\Psi^\dagger e^{qV^a T_a} \Psi$, to be invariant under non-abelian gauge transformations. Now, using that we write the generators t_a as hermitian operators,

$$\Psi^\dagger e^{qV^a T_a} \Psi \rightarrow \Psi'^\dagger e^{qV'^a T_a} \Psi' = \Psi^\dagger e^{iq\Lambda^{a\dagger} T_a} e^{qV'^a T_a} e^{-iq\Lambda^a T_a} \Psi,$$

so we have to require that the vector superfield V transforms as⁸

$$e^{qV'^a T_a} = e^{-iq\Lambda^{a\dagger} T_a} e^{qV^a T_a} e^{iq\Lambda^a T_a}. \quad (4.14)$$

When we look at this as an infinitesimal transformation in Λ we can show that

$$V'^a = V^a + i(\Lambda^a - \Lambda^{a\dagger}) - \frac{1}{2} q f_{bc}^a V^b (\Lambda^{c\dagger} + \Lambda^c) + \mathcal{O}(\Lambda^2),$$

⁶By unitary we mean, as usual, that $U^\dagger = U^{-1}$ so that $U^\dagger U = 1$.

⁷Of, course, you may ask, how do we even know that we can find a unitary representation for a particular Lie group? For the unitary group this is by definition, but is it possible in general? It turns out that this is always true for a subset of Lie groups, called **compact Lie groups**. These are the Lie groups where the parameters vary over a closed interval.

⁸This is independent of our choice of representation for the gauge group for the supergauge transformation.

which reduces to the definition for abelian groups in (3.28) since all the higher order terms contain the structure constant f_{bc}^a , which is zero for abelian groups. If we look at the component vector fields, V_μ^a , these transform just like in a standard non-abelian gauge theory:

$$V_\mu^a \rightarrow V_\mu'^a = V_\mu^a + i\partial_\mu(A^a - A^{a*}) - qf_{bc}^a V_\mu^b (A^c - A^{c*}),$$

in the adjoint representation of the gauge group.

The supergauge transformations of vector superfields can be written more efficiently in a representation independent way as

$$e^{V'} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda},$$

and the inverse transformation is then given by

$$e^{-V'} = e^{-i\Lambda} e^{-V} e^{i\Lambda^\dagger},$$

where $\Lambda \equiv q\Lambda^a T_a$ and $V \equiv qV^a T_a$, such that $e^V e^{-V} = e^{V'} e^{-V'} = 1$.⁹

4.4 Supersymmetric field strength

There is one type of term missing from the supersymmetric Lagrangian we are constructing, namely field strength terms that make the gauge fields dynamical, *e.g.* terms to describe the electromagnetic field strength $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ in QED.

Definition: The **supersymmetric field strength** for a vector superfield V is defined by the spinor (matrix) scalar superfields W_A and $\bar{W}_{\dot{A}}$ given by

$$W_A \equiv -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_A e^V,$$

and

$$\bar{W}_{\dot{A}} \equiv -\frac{1}{4}DDe^{-V}\bar{D}_{\dot{A}} e^V,$$

where $V = V^a T_a$.

For an abelian gauge field this definition reduces to

$$\begin{aligned} W_A &= -\frac{1}{4}\bar{D}\bar{D}D_A V, \\ \bar{W}_{\dot{A}} &= -\frac{1}{4}DD\bar{D}_{\dot{A}} V, \end{aligned}$$

where V is simply the vector superfield of the gauge group.

We can show that the components of W_A ($\bar{W}_{\dot{A}}$) are left-handed (right-handed) scalar superfields, and that both $\text{Tr}[W^A W_A]$ and $\text{Tr}[\bar{W}_{\dot{A}} \bar{W}^{\dot{A}}]$ are supergauge invariant constructions, and thus potential terms in the supersymmetry Lagrangian. Firstly

$$\bar{D}_{\dot{A}} W_A = -\frac{1}{4}\bar{D}_{\dot{A}}\bar{D}\bar{D}e^{-V}D_A e^V = 0,$$

⁹Notice that despite the non-commutative nature of the matrices involved, the identity $e^A e^{-A} = 1$ holds. See Exercise 11.19.

because from Eq. (3.16), $\bar{D}^3 = 0$, and similarly for $\bar{W}_{\dot{A}}$, showing that they are both superfields, and as such can be used to form a supersymmetric Lagrangian.

Under a supergauge transformation we have:

$$\begin{aligned}
W_A \rightarrow W'_A &= -\frac{1}{4}\bar{D}\bar{D}e^{-i\Lambda}e^{-V}e^{i\Lambda^\dagger}D_Ae^{-i\Lambda^\dagger}e^Ve^{i\Lambda} \\
(\bar{D}_{\dot{A}}\Lambda = 0) &= -\frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}e^{-V}e^{i\Lambda^\dagger}D_Ae^{-i\Lambda^\dagger}e^Ve^{i\Lambda} \\
(D_A\Lambda^\dagger = 0) &= -\frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}e^{-V}D_Ae^Ve^{i\Lambda} \\
&= -\frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}e^{-V}[(D_Ae^V)e^{i\Lambda} + e^V(D_Ae^{i\Lambda})] \\
&= e^{-i\Lambda}W_Ae^{i\Lambda} - \frac{1}{4}e^{-i\Lambda}\bar{D}\bar{D}D_Ae^{i\Lambda}.
\end{aligned} \tag{4.15}$$

We are free to add zero to (4.15) in the form of $-\frac{1}{4}e^{-i\Lambda}\bar{D}D_A\bar{D}e^{i\Lambda} = 0$ ^[10] giving

$$\begin{aligned}
W'_A &= e^{-i\Lambda}W_Ae^{i\Lambda} - \frac{1}{4}e^{-i\Lambda}\bar{D}\{\bar{D}, D_A\}e^{i\Lambda} \\
&= e^{-i\Lambda}W_Ae^{i\Lambda} + \frac{1}{2}e^{-i\Lambda}\bar{D}_{\dot{A}}\sigma^\mu_{\dot{A}\dot{B}}\epsilon^{\dot{A}\dot{B}}P_\mu e^{i\Lambda} \\
&= e^{-i\Lambda}W_Ae^{i\Lambda},
\end{aligned}$$

where we have used Eq. (3.15) to replace the anti-commutator. This means that the following trace is gauge invariant:

$$\begin{aligned}
\text{Tr}[W'^AW'_A] &= \text{Tr}[e^{-i\Lambda}W^Ae^{i\Lambda}e^{-i\Lambda}W_Ae^{i\Lambda}] \\
&= \text{Tr}[e^{i\Lambda}e^{-i\Lambda}W^AW_A] = \text{Tr}[W^AW_A],
\end{aligned}$$

and can be used in the Lagrangian.

If we expand W_A in the component fields we find, as we might have hoped, that it contains the ordinary field strength tensor:

$$F_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + qf_{bc}^a V_\mu^b V_\nu^c$$

and that the trace indeed contains terms with $F_{\mu\nu}^a F^{\mu\nu a}$.

4.5 The (almost) complete supersymmetric Lagrangian

We can now compile all our results up to now to write down the complete Lagrangian for a supersymmetric theory with (possibly) non-abelian gauge symmetries.^[11]

$$\mathcal{L} = \Phi^\dagger e^V \Phi + \delta^2(\bar{\theta})W[\Phi] + \delta^2(\theta)W[\Phi^\dagger] + \frac{1}{2T(R)}\delta^2(\bar{\theta})\text{Tr}[W^AW_A]. \tag{4.16}$$

¹⁰Which is zero because Λ is a left-handed scalar superfield, $\bar{D}_{\dot{A}}\Lambda = 0$.

¹¹There is no hermitian conjugate of the field strength term, and a slightly odd normalisation using $1/2$ instead of $1/4$. This is because the term can be proven to be real, although this is sometimes overlooked in the literature when authors instead use

$$\mathcal{L} \sim \frac{1}{4T(R)}\delta^2(\bar{\theta})\text{Tr}[W^AW_A] + \frac{1}{4T(R)}\delta^2(\theta)\text{Tr}[\bar{W}^A\bar{W}_A],$$

which looks more symmetric with respect to the superpotential part.

Here, $T(R)$ given by $T(R)\delta_{ab} = \text{Tr}[T_a T_b]$ is called the **Dynkin index** of the representation R of the gauge group using the generators T_a .

The Dynkin index appears in the field strength term to correctly normalise the energy density for the chosen representation R of the gauge group. To see that this factor cancels in a natural way, note that since W_A is spanned by T_a for a given representation, we can write $W_A = W_A^a T_a$, where W_A^a are some coefficients. Then

$$\text{Tr}[W^A W_A] = W^{aA} W_A^b \text{Tr}[T_a T_b] = W^{aA} W_A^b \delta_{ab} T(R) = T(R) W^{aA} W_A^a. \quad (4.17)$$

4.6 Spontaneous supersymmetry breaking

As we have seen above, supersymmetry predicts scalar partner particles with the same mass as the known fermions (and new fermions for the known vectors). These, somewhat unfortunately, contradict experiment by not existing. In the SM we have a similar problem: the vector bosons should remain massless under the gauge symmetry of the model. Yet, they are observed to be very massive. This is solved with the introduction of the Higgs mechanism and **spontaneous symmetry breaking** in the **scalar potential**.^[12] The idea is that while there is a symmetry of the Lagrangian (in the SM the gauge symmetry), this may not be a symmetry of the vacuum state, thereby allowing the properties of the vacuum to supply the masses. Would it not be great if we could have *spontaneous symmetry breaking* in order to break supersymmetry this way and boost the masses of supersymmetric particles beyond current limits?

From exercise [1](#) we can see that the Lagrangian of [\(4.16\)](#) written in terms of component field contains no kinetic (derivative) terms for the $F(x)$ scalar fields. These are then what we call **auxiliary fields** and can be eliminated by the e.o.m. we get from solving the Euler-Lagrange equation for this field:^[13]

$$\frac{\partial \mathcal{L}}{\partial F_i^*(x)} = F_i(x) + W_i^* = 0,$$

where

$$W_i \equiv \frac{\partial W[A_1, \dots, A_n]}{\partial A_i}. \quad (4.19)$$

This allows us to rewrite the action as (ignoring gauge interactions):

$$S = \int d^4x \{ i \partial_\mu \bar{\psi}_i \sigma^\mu \psi_i - A_i^* \square A_i - \frac{1}{2} W_{ij} \psi_i \psi_j - \frac{1}{2} W_{ij}^* \bar{\psi}_i \bar{\psi}_j - |W_i|^2 \}$$

with [\[14\]](#)

$$W_{ij} \equiv \frac{\partial^2 W[A_1, \dots, A_n]}{\partial A_i \partial A_j}. \quad (4.20)$$

^[12]The potential of the Lagrangian are those terms not containing derivatives of the fields (kinetic terms). The scalar potential are such terms that contain only scalar fields.

^[13]We remind the reader that the Euler-Lagrange equation for a field ϕ is the result of minimizing the action and is given in terms of the Lagrangian as:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0. \quad (4.18)$$

^[14]This is called the **fermionic mass matrix**.