

For a concrete example of how this can be worked out, see Exercise 4.1. Generalising this also leads to an expression for the action for any number of scalar superfields Φ_i , ignoring gauge interactions, which is:

$$S = \int d^4x \{ i \partial_\mu \bar{\psi}_i \sigma^\mu \psi_i - A_i^* \square A_i - \frac{1}{2} W_{ij} \psi_i \psi_j - \frac{1}{2} W_{ij}^* \bar{\psi}_i \bar{\psi}_j - |W_i|^2 \}$$

where W_{ij} , given by

$$W_{ij} \equiv \frac{\partial^2 W[A_1, \dots, A_n]}{\partial A_i \partial A_j}, \quad (4.20)$$

is called the **fermionic mass matrix**.

The D^a -fields occur in the kinetic terms and the field strength terms. In the kinetic term, in the Wess–Zumino gauge, it is relatively easy to see that D^a appears only once in the expansion of the exponential as $q A_i^* D^a T_{ij}^a A_j$ in the action. In the field strength term one can show with some more effort that the contribution is $D^a D^a$ in the Wess–Zumino gauge. For abelian gauge fields there is also a possible kD contribution from the Fayet–Iliopoulos term. This leads to the Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial D^a} = q A_i^* T_{ij}^a A_j + 2D^a = 0,$$

which can be used to eliminate D^a .

Note that the terms discussed here with F - and D -fields exhaust all the non-derivative terms that only have scalar fields since no terms with only A -fields can survive the θ -integration. Thus the complete non-derivative scalar field contribution to the Lagrangian, called the **scalar potential**, can be written as

$$V(A_i, A_i^*) = \sum_i |F_i|^2 + \sum_a D^a D^a = \sum_i \left| \frac{\partial W[A_1, \dots, A_n]}{\partial A_i} \right|^2 + \frac{1}{4} \sum_a q^2 (A_i^* T_{ij}^a A_j)^2. \quad (4.21)$$

4.7 The hierarchy problem

There is a fundamental problem with having scalar particles in a quantum field theory. Let us take the Standard Model Higgs boson h as an example, however, the following would be true for any scalar particle. If we naively calculate loop-corrections to its mass in self-energy diagrams like the ones shown in Fig. 4.1, where f is a fermion and s some other scalar, they both diverge due to their loop momenta integrals, meaning they are infinite. This then needs what is called **regularisation** in order to yield a finite answer.

There are different ways of achieving regularisation. Since we know that the Standard Model is an incomplete theory, at least when we go up to Planck scale energies where we need an unknown quantum theory of gravity, we can introduce a **cut-off regularisation** where we limit the integral in the loop-correction to momenta below a scale Λ_{UV} . Then the loop-correction to the Higgs mass is, at leading order in Λ_{UV} ,

$$\Delta m_h^2 = -\frac{|\lambda_f|^2}{8\pi^2} \Lambda_{UV}^2 + \frac{\lambda_s}{16\pi^2} \Lambda_{UV}^2 + \dots \quad (4.22)$$

where λ_f and λ_s are the couplings of f and s to the Higgs, respectively. Picking Λ_{UV} suggestively as the Planck scale, $\Lambda_{UV} = M_P = 2.44 \times 10^{18}$ GeV, meaning that the quantum

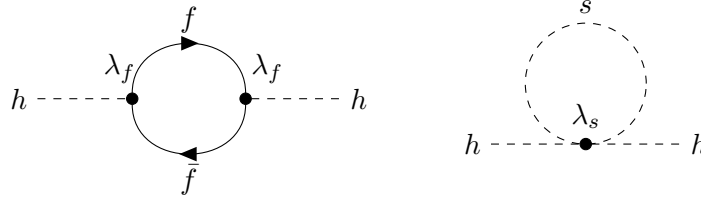


Figure 4.1: Higgs self-energy diagrams with fermions f (left) and scalars s (right).

corrections to the Higgs mass are some 15 orders of magnitude greater than its measured value.

We observe that the difference in sign between the fermions and bosons in (4.22) means that it could *in principle* be possible that these huge contributions cancel to keep $m_h \sim 125 \text{ GeV}$ as measured, however, as physicists we should worry why the Universe seems to be organised in such a strange way without any specific cause. This is known as the **Higgs fine tuning problem**, or the **hierarchy problem** because it is fundamentally a problem of why the electroweak scale at around 100 GeV where the Higgs lives, is so much lighter than the Planck scale of 10^{18} GeV , when the former should in principle be pulled up to the latter by the sensitivity of scalars to loop corrections.¹⁴

Enter supersymmetry to the rescue: with unbroken supersymmetry we find that we automatically have $|\lambda_f|^2 = \lambda_s$ and exactly twice as many scalar as fermion degrees of freedom running around in loops. This provides a magic cancellation of the quadratic divergence in Eq. (4.22). To see that this relation between the couplings holds, notice that it is the superpotential term $W \sim \lambda_{ijk} \Phi_i \Phi_j \Phi_k$ which is responsible for generating Lagrangian Yukawa terms of the form $\lambda_{ijk} \psi_i \psi_j A_k$ that couple two fermions to a scalar. At the same time, in the scalar potential, that same superpotential term is responsible for the terms

$$V(A, A^*) \sim \left| \frac{\partial W}{\partial A_i} \right|^2 = |\lambda_{ijk}|^2 A_j^* A_k^* A_j A_k. \quad (4.23)$$

When the scalar field A_k is the Higgs field, the loop fermion is represented by $\psi_i = \psi_j$ and the second scalar by A_j , these two terms are responsible for the two types of vertices in Fig. 4.1 with $\lambda_f = \lambda_{ijk}$ and $\lambda_s = |\lambda_{ijk}|^2$. The fact that there are loop contributions from two scalars for each fermion due to the same number of states (two scalar states per fermion state), means that unbroken supersymmetry predicts an exact cancellation in Eq. (4.22).¹⁵

4.8 Vacuum energy

This measured value of the dark energy in terms of an energy scale is $\Lambda_{DE} \simeq 2.4 \times 10^{-3} \text{ eV}$. This can be interpreted as vacuum energy, *i.e.* the energy predicted from the contribution of Feynman diagrams with *no* external particles. However, the predicted value of the vacuum

¹⁴With some background in renormalisation you may ask: What about choosing dimensional regularisation instead where there is no cut-off scale? That would in principle work, however, as soon as you introduce *any* new particle (significantly) heavier than the Higgs there is still a quadratic correction of the form of (4.22) with the new particle masses as the scale, meaning that we cannot complete the Standard Model at a higher scale without reintroducing the problem.

¹⁵To keep the argument simple we have avoided the contributions from vector bosons, however, we can show that these also cancel exactly against the contributions from their fermionic partners.

energy in the Standard Model is $\Lambda_{VE} \sim M_P$.^[16] If we want to compare energy densities, we should compare Λ_{DE}^4 to Λ_{VE}^4 ,^[17] which means that the prediction is off from the measurement by some 120 orders of magnitude, which is said to be a record for the greatest ever discrepancy between theory and experiment. This problem is the **hierarchy problem for vacuum energy**.

Can supersymmetry save us here as well? Sort of. For an unbroken global supersymmetry we can use the **non-renormalisation theorem**^[18] of Grisaru, Roach and Siegel (1979) [10] to show that the prediction in supersymmetry is exactly $\Lambda_{VE} = 0$.

Theorem: Non-renormalisation theorem

All higher order contributions to the effective supersymmetric action for a process S_{eff} can be written:

$$S_{\text{eff}} = \sum_n \int d^4x_i \dots d^4x_n d^4\theta F_1(x_1, \bar{\theta}, \theta) \times \dots \times F_n(x_1, \bar{\theta}, \theta) \times G(x_1, \dots, x_n), \quad (4.24)$$

where F_i are products of the n external superfields and their covariant derivatives, and G is a supersymmetry invariant function.

While this is a rather technical statement, that fact that vacuum diagrams do not have external superfields means that the higher order contributions are all zero, and since there are no tree-level contributions for vacuum diagrams – you must have loops to not have external legs – there can be no contribution at all.

This was a victory for supersymmetry before the discovery of dark energy, when the problem was instead to prove that $\Lambda = 0$, modulo the fact that the breaking supersymmetry, as we will be doing in Chapter 5, changes this prediction. As we shall see there, the scale of this contribution has to be the mass scale of the supersymmetric particles, so with for example $m_{SUSY} \simeq 2 \text{ TeV}$, we have $m_{SUSY}/\Lambda_{DE} \simeq 10^{15}$, some 15 orders of magnitude too large, which is better than the Standard Model prediction of $M_P/\Lambda_{DE} = 10^{30}$, but still a bit off the measured value.^[19]

So now we are left with showing that the contribution is very small but non-zero, which is in general though to be a much harder problem than finding models where it is exactly zero. However, in supergravity something interesting happens. Introducing a local supersymmetry the scalar potential is not simply given by the superpotential derivatives in (4.21), but instead

^[16]The origin of this is just the same as the quadratic divergence for the Higgs mass. It is the same type of diagrams contributing, only without external legs.

^[17]In natural units the unit of energy density is $[\rho] = [E/V] = M/L^3 = M^4$.

^[18]The name of this theorem is a bit funny. Why? Well, mainly because the result is not about not being able to renormalise the theory, but about about not *needing* to renormalise certain parts of it. Another consequence of the theorem, which gives it its name, is that the couplings of the superpotential do not need separate renormalisation. The renormalisation of the superfields suffices.

^[19]So, one would be tempted to say that supersymmetry has solved half the problem. On a more serious note, there is a significant difference in that the Standard Model prediction requires a cut-off to be finite, while supersymmetry predicts a finite but too large value.

is (ignoring the effects of gauge fields)

$$V(A, A^*) = e^{K/M_P} \left[K_{ij}(D_i W)(D_j W^*) - \frac{3}{M_P^2} |W|^2 \right], \quad (4.25)$$

where $K_{ij} = \partial_i \partial_j K[A, A^*]$ is the **Kähler metric**, D_i the **Kähler derivative** $D_i = \partial_i + \frac{1}{M_P^2}(\partial_i K)$, and all the derivatives are with respect to the scalar fields in the Kähler potential K and superpotential W . In the $M_P \rightarrow \infty$ limit, the low energy limit, we see that we recover the flat space result of Eq. (4.21). What is important to notice is that there is now a second *negative* term in the potential that can in principle cancel the positive supersymmetry breaking contribution, however, given the large size of the breaking contribution this will come at the price of fantastic fine-tuning unless some mechanism can be found where this is natural.

4.9 Exercises

Exercise 4.1

Write down the Lagrangian and find the action of the simplest possible supersymmetric field theory with a single scalar superfield, without gauge transformations, in terms of component fields, and show that it contains no kinetic terms for the F -field. Then show how the F -field can be eliminated by the equations of motion. *Hint:* The kinetic part of the action will turn out to be

$$S_{\text{kin}} = \int d^4x \left\{ -A^*(x) \square A(x) + |F(x)|^2 + i \partial_\mu \psi(x) \bar{\sigma}^\mu \psi(x) \right\}. \quad (4.26)$$

To show this you may have use of the identities in Sec. 2.4.1.

Exercise 4.2

Extend the previous exercise to a single abelian gauge group. *Hint:* It may be productive to consider the coordinate change $y^\mu = x^\mu + i\theta \sigma^\mu \theta$ for the field strength term.

Exercise 4.3

Find the Lagrangian for supersymmetric QED (SQED) in terms of component fields. Supersymmetric QED is the smallest (in terms of field content) supersymmetric theory that has the particles and interactions in QED as a subset.

Chapter 5

Breaking supersymmetry

In Chapter 2 we saw that supersymmetry predicts scalar partner particles with the same mass as the known fermions, and new fermions with the same mass as the known vectors, living in the same irreducible representation. These, somewhat unfortunately, contradict experiment by not existing. In this chapter we will look at how we can break supersymmetry so that the new states become more massive, and avoid current experimental bounds. This discussion will be held at a general level, and concrete realisations of this breaking will have to wait until Chapter 6.

5.1 Spontaneous supersymmetry breaking

In the Standard Model we have already faced this problem: the vector bosons should remain massless under the gauge symmetry of the model. Yet, they are observed to be very massive. This is solved with the introduction of the Higgs mechanism and **spontaneous symmetry breaking** in the **scalar potential**.¹ The idea is that while there is a symmetry of the Lagrangian – in the Standard Model the gauge symmetry $SU(3)_c \times SU(2)_L \times U(1)_Y$ – this may not be a symmetry for the particular vacuum state of the theory (the lowest energy state), thereby allowing the properties of the vacuum to supply the masses. In the Standard Model this is achieved by the shape of the scalar potential having a minimum for a non-zero value of the Higgs field which means that the value of the field there – its vacuum expectation value – can contribute to the masses. Would it not be great if we could have a similar *spontaneous symmetry breaking* in order to break supersymmetry this way and boost the masses of supersymmetric particles beyond current limits?

To find the properties of the vacuum state for our supersymmetric models we start by pointing out that we can write the supersymmetric Hamiltonian as

$$H = \frac{1}{4}(Q_1\bar{Q}_1 + \bar{Q}_1Q_1 + Q_2\bar{Q}_2 + \bar{Q}_2Q_2).$$

To see this, consider first

$$\begin{aligned} \{Q_A, \bar{Q}_{\dot{B}}\}\bar{\sigma}^{\nu\dot{B}A} &= 2\sigma^\mu_{A\dot{B}}\bar{\sigma}^{\nu\dot{B}A}P_\mu \\ &= 2\text{Tr}[\sigma^\mu\bar{\sigma}^\nu]P_\mu \\ &= 4g^{\mu\nu}P_\mu = 4P^\nu. \end{aligned}$$

¹The potential of the Lagrangian are those terms not containing derivatives of the fields (kinetic terms). The scalar potential are such terms that contain only scalar fields.

Now,

$$\begin{aligned} H &= P^0 = \frac{1}{4}\{Q_A, \bar{Q}_{\dot{B}}\}\bar{\sigma}^{0\dot{B}A} \\ &= \frac{1}{4}(Q_1\bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}}Q_1 + Q_2\bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}}Q_2). \end{aligned}$$

As discussed in Section 2.4 we have $Q_A^\dagger = \bar{Q}_{\dot{A}}$. Thus the Hamiltonian is **semipositive definite**, *i.e.* $\langle\Psi|H|\Psi\rangle \geq 0$ for any state $|\Psi\rangle$, so the energy of any state is always non-negative.

Imagine now that there exists some lowest lying state (or possibly degenerate states), the ground state(s) $|0\rangle$, that has vanishing energy $\langle 0|H|0\rangle = 0$. These states are supersymmetric – meaning invariant under the supersymmetry transformation – since, to fulfil the energy assumption, we must have

$$Q_A|0\rangle = \bar{Q}_{\dot{A}}|0\rangle = 0 \quad \text{for } \forall A, \dot{A}, \quad (5.1)$$

and are thus invariant under the supersymmetry transformations given by (4.3)

$$\delta_S|0\rangle = (\alpha^A Q_A + \bar{\alpha}_{\dot{A}} \bar{Q}^{\dot{A}})|0\rangle = 0. \quad (5.2)$$

This means that at this supersymmetric minimum of the potential the scalar potential must contribute zero, $V(A, A^*) = 0$, and thus, from Eq. (4.21),

$$\left\langle \frac{\partial W}{\partial A_i} \right\rangle = 0 \quad \text{and} \quad \langle A_i^* T_{ij}^a A_j \rangle = 0.$$

Conversely, if the scalar potential does contribute in the vacuum (ground state) $|0\rangle$, meaning

$$\left\langle \frac{\partial W}{\partial A_i} \right\rangle \neq 0 \quad \text{or} \quad \langle A_i^* T_{ij}^a A_j \rangle \neq 0,$$

in the minimum of the potential for some A_i , then *supersymmetry must be broken*. As in the Standard Model, the Lagrangian is still (super)symmetric, but $|0\rangle$ is not because (5.1) can no longer hold for all the Q s.

The **O’Raifeartaigh model** (1975) [8] is an example of a model that spontaneously breaks supersymmetry with three scalar superfields X , Y , Z , and the superpotential

$$W = \lambda Y Z + g X (Z^2 - m^2), \quad (5.3)$$

where λ , g and m are real non-zero parameters. The scalar potential is

$$\begin{aligned} V(A, A^*) &= \left| \frac{\partial W}{\partial A_X} \right|^2 + \left| \frac{\partial W}{\partial A_Y} \right|^2 + \left| \frac{\partial W}{\partial A_Z} \right|^2 \\ &= |g(A_Z^2 - m^2)|^2 + |\lambda A_Z|^2 + |\lambda A_Y + 2g A_X A_Z|^2, \end{aligned} \quad (5.4)$$

which can never be zero because setting $A_Z = 0$, which is needed for the second term, gives a non-zero contribution $g^2 m^4$ from the first term. Since the expectation value at the minimum that breaks supersymmetry is $\langle 0 | \frac{\partial W_i}{\partial A_i} | 0 \rangle$, and $F_i = -\frac{\partial W_i^*}{\partial A_i}$, the condition for spontaneous supersymmetry breaking with the O’Raifeartaigh mechanism can be written

$$\langle F_i \rangle \equiv \langle 0 | F_i(x) | 0 \rangle > 0, \quad (5.5)$$

hence it is often given the more generic name **F-term breaking** outside of the specific O’Raifeartaigh model. In general F -term breaking it is the **vacuum expectation value** (**vev**) of the auxiliary field of a scalar superfield that supplies the breaking.

In a gauge theory, a similar mechanism is found by adding the Fayet-Iliopoulos term $\mathcal{L}_{FI} \sim 2kV$ where V is an abelian vector superfield. The vev of the $D(x)$ auxiliary field, $\langle D \rangle = \langle 0|D(x)|0 \rangle$, will then create a non-zero scalar potential.^[2] This is called the **Fayet-Iliopoulos model**, or **D-term breaking**.

5.2 Supertrace

Unfortunately, the above does not work in practice with all particles at a low energy scale. The problem is that at *tree level* the **supertrace**, STr , the weighted sum of eigenvalues of the mass matrix $\mathcal{M}$, can be shown to vanish, $\text{STr } \mathcal{M}^2 = 0$.^[3]

Definition: The **supertrace** is given by

$$\text{STr } \mathcal{M}^2 \equiv \sum_s (-1)^{2s} (2s+1) \text{Tr } M_s^2 \quad (5.6)$$

where $\mathcal{M}$ is the mass matrix of the Lagrangian, s is the spin of particles and M_s is the mass matrix of all spin- s particles.

For a theory with only scalar superfields, with two fermionic and two bosonic degrees of freedom each, and with, respectively, mass matrices $M_{1/2}$ and M_0 after spontaneous supersymmetry breaking, this means that $\text{Tr } \{M_0^2 - 2M_{1/2}^2\} = 0$, *i.e.* the sum of scalar particle masses (squared) is equal to the fermion masses (squared).^[4] The consequence is that not all the scalar partners can be heavier than our known fermions.^[5]

5.3 Soft breaking

What we can do instead is to add explicit supersymmetry breaking terms to the Lagrangian parametrizing our ignorance of the true (spontaneous) supersymmetry breaking on some higher scale $\sqrt{\langle F \rangle}$ that we do not have access to where the supertrace relation is fulfilled,^[6] for which there are many alternatives in the literature, *e.g.*:

- Planck-scale Mediated Symmetry Breaking (PMSB)
- Gauge Mediated Symmetry Breaking (GMSB)
- Anomaly Mediated Symmetry Breaking (AMSB)

However, we cannot simply add arbitrary terms to the Lagrangian. The terms we can add are so-called **soft terms** with couplings of mass dimension one or higher. The dis-allowed terms

²It is *always* the auxiliary fields’ fault!

³See Ferrara, Girardello and Palumbo (1979) [9].

⁴Remember that there are two scalar particles for each fermion.

⁵Strong coupling, meaning tree level is a bad approximation, may help, but life is still difficult.

⁶Remember that $[\Phi] = M$ and $[\theta] = M^{-\frac{1}{2}}$ so that the component field must have $[F] = M^2$.