

Chapter 1

Groups and algebras

The study of symmetries plays a central part in theoretical physics, and the mathematical language we use is that of groups. The action of the group elements on our (quantum mechanical) states describes the transformation of the symmetry operation, while the invariance of the physical properties of the system under that transformation is the symmetry itself. For example, rotations in three-dimensions can be carried out by the application of a 3×3 rotation matrix on the coordinates. These matrices form the group called $SO(3)$. For a sphere, which is invariant under these rotations, $SO(3)$ is then the symmetry group.

Of special interest to us are the Lie groups, which are the groups that represent continuous transformations, such as the rotations. The properties of Lie groups can be further studied by finding their generators which form a (Lie) algebra. The generators *almost* – in a very specific sense of the word almost – describes the whole group, and allows us to reconstruct the group elements by what is called the exponential map.

Here, we will begin by defining groups and looking at some of their most important properties. What is crucial in physics are the *representations* of groups, meaning what the operators of the transformations on the states actually look like. Returning to the rotation example these are 3×3 matrices, but with some restrictions on their elements. After discussing representations we will move on to defining Lie groups, before we end on a discussion of their generators and corresponding algebras.

1.1 Group definition

A group is an abstract mathematical structure that consists of a set of objects (elements), and a multiplication rule acting between pairs of these objects. We define a group as follows.

Definition: The set of elements $G = \{g_i\}$ and operation $\circ$ (sometimes called multiplication) form a **group** if and only if for $\forall g_i \in G$:

$$\text{i) } g_i \circ g_j \in G, \quad (\text{closure})$$

$$\text{ii) } (g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k), \quad (\text{associativity})$$

$$\text{iii) } \exists e \in G \text{ such that } g_i \circ e = e \circ g_i = g_i, \quad (\text{identity element})$$

$$\text{iv) } \exists g_i^{-1} \in G \text{ such that } g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = e. \quad (\text{inverse})$$

Below, where no confusion can occur, we will often drop the multiplication symbol for the group multiplication (and other abstract multiplications), writing $g_i \circ g_j = g_i g_j$.

A straight forward example of a group is $G = \mathbb{Z}$ (the integers), with standard addition as the operation $\circ$. Then $e = 0$ and $g^{-1} = -g$. Alternatively we can restrict the group to $\mathbb{Z}_n$, where the operation is addition modulo n . In this group, $g_i^{-1} = n - g_i$ and the unit element is again $e = 0$.^[1] Here, $\mathbb{Z}$ is an example of an **infinite** group, the set has an infinite number of members, while $\mathbb{Z}_n$ is **finite**, with **order** n , meaning n members. Both are **abelian** groups, meaning that the elements commute: $g_i \circ g_j = g_j \circ g_i$, because the standard addition commutes.

The simplest, non-trivial, of these $\mathbb{Z}_n$ -groups is $\mathbb{Z}_2$ which only has the members $e = 0$ and 1. The “multiplication” operation is completely defined by the three possibilities $0 + 0 = 0$, $0 + 1 = 1$ and $1 + 1 = 0$. Now, compare this to the set $G = \{-1, 1\}$ with the ordinary multiplication operation. Here, all the possible operations are $1 \cdot 1 = 1$, $1 \cdot (-1) = -1$ and $(-1) \cdot (-1) = 1$. This has exactly the same structure as $\mathbb{Z}_2$, only that the identity element is now 1. We say that these two groups are **isomorphic**, because there is a mapping between all the (two) elements $0 \rightarrow 1$ and $1 \rightarrow -1$, and the results of the multiplication operation is the same, and in fact we consider them as the *same group* despite the considerable apparent visual differences.^[2] This notion of isomorphic (identical) groups is very important, and we will return to it in more detail in Sec. 1.3.

A somewhat more sophisticated example of a group can be found in the Taylor expansion of a function f , where

$$\begin{aligned} f(x+a) &= f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \dots \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n}{dx^n} f(x) \\ &= e^{a \frac{d}{dx}} f(x). \end{aligned}$$

The last equality uses the formal definition of the exponential series, but may drive some mathematicians crazy.^[3] The resulting operator $T_a = e^{a \frac{d}{dx}}$ is called the **translation operator**,

¹Note that we here use e for the identity in an abstract group, while we will later use I or 1 as the identity matrix in matrix representations of groups.

²This observation generalises to the set $G = \{e^{2\pi i k/n} | k = 1, \dots, n-1\}$, the **n -th roots of unity**, which, together with the standard multiplication operation, is isomorphic to $\mathbb{Z}_n$.

³We will not discuss this further, but there is a deep question here whether the operator formed by this exponentiation is well defined.

in this case in one dimension, since it shifts the coordinate x of the function f it is operating on by an amount a . Defining the (natural) multiplication operation $T_a \circ T_b = T_{a+b}$ it forms the **translational group** $T(1)$, where we can show that $T_a^{-1} = T_{-a}$.⁴ In n dimensions the group $T(n)$ has the elements $T_{\mathbf{a}} = e^{\mathbf{a} \cdot \nabla}$. Whereas we say that the groups $\mathbb{Z}$ and $\mathbb{Z}_n$ are **discrete groups**, since we can count the number of elements, T_a is a **continuous group** since the parameter a can be any real number.

1.2 Matrix groups

We next define some groups that are very important in physics and to the discussion in these notes. They have in common that they are defined in terms of square matrices.

1.2.1 General and special linear groups

The largest matrix group for a given matrix dimension is the general linear group.

Definition: The **general linear group** $GL(n)$ is defined by the set of invertible $n \times n$ matrices A under matrix multiplication. If we additionally require that $\det(A) = 1$, the matrices form the **special linear group** $SL(n)$.

The existence of the group identity is guaranteed by the identity matrix I being an invertible matrix (with I as the inverse). Since the existence of an inverse is also necessary in the group definition, we can not construct larger matrix groups. The general linear group also give us our first example of a **non-abelian group**, since matrix multiplication does not in general commute. For two matrices A and B , we may have $AB \neq BA$.

We usually take the matrices in matrix groups to be defined over the field of complex numbers $\mathbb{C}$. If we want to specify the field we may use the notation $GL(n, \mathbb{R})$, signifying that the group is defined over the real numbers. Defined over the complex numbers the $GL(n)$ groups have $2n^2$ free parameters since each of the n^2 elements of the matrices can be a complex number, needing two parameters. The $SL(n)$ group has $2n^2 - 2$ free parameters since the requirement on the determinant fixes both the real and imaginary part of the determinant.

1.2.2 Unitary and special unitary groups

We first remind you that the **Hermitian conjugate** or conjugate transpose of a matrix is given by transposing the matrix and taking the complex conjugate of its elements. Here, we will use the symbol $\dagger$ for this operation, so that for a matrix A , $A^\dagger = (A^T)^*$.

We now define the unitary groups.

Definition: The **unitary group** $U(n)$ is defined by the set of complex unitary $n \times n$ matrices U , *i.e.* matrices such that $U^\dagger U = I$ or $U^{-1} = U^\dagger$. If we additionally require that $\det(U) = 1$ the matrices form the **special unitary group** $SU(n)$.

⁴We could instead have defined the operation between two group elements to be ordinary multiplication and used that to show the relationship $T_a \circ T_b = T_{a+b}$. However, it is important to notice that showing this is not entirely trivial because ordinary arithmetic rules for exponentials fail for operators. In this particular case the proof is fairly simple, but this is in general not so.

Since for $U \in U(n)$,

$$\det(UU^\dagger) = \det(U) \det(U^\dagger) = \det(U) \det(U^T)^* = \det(U) \det(U)^* = \det(I) = 1,$$

we have that the determinant of these matrices must be complex numbers on the unit circle, *i.e.* $\det(U) = e^{i\theta}$. It can be shown (see exercises) that the $U(n)$ groups have n^2 independent parameters, while the $SU(n)$ groups have $n^2 - 1$.

It is these groups that form the gauge symmetry groups of the Standard Model: $SU(3)$, $SU(2)$ and $U(1)$. The group $U(1)$ makes perfect sense despite the odd matrix dimensions. This is simply the set of all complex numbers of unit length with ordinary multiplication, *i.e.* $U(1) = \{e^{i\alpha} | \alpha \in \mathbb{R}\}$, but notice that $SU(1)$ would be trivial since it contains only 1.

The unitary group has the important property that for $\forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ multiplication by a unitary matrix leaves scalar products unchanged. If $\mathbf{x}' = U\mathbf{x}$ and $\mathbf{y}' = U\mathbf{y}$, then

$$\begin{aligned} \mathbf{x}' \cdot \mathbf{y}' &\equiv \mathbf{x}'^\dagger \mathbf{y}' = (U\mathbf{x})^\dagger U\mathbf{y} \\ &= \mathbf{x}^\dagger U^\dagger U \mathbf{y} = \mathbf{x}^\dagger \mathbf{y} = \mathbf{x} \cdot \mathbf{y}. \end{aligned}$$

Thus, its members do not change the length of the vectors they act on.

Since we would like to let our group representations act on vectors that describe quantum mechanical states, the unitary groups then conserve probability for these states. For example, when acting on a complex number (a complex scalar), such as a wavefunction $\psi(x)$, the elements of $U(1)$ rotate the phase of ψ , however, the magnitude is conserved since $\psi' = e^{i\alpha}\psi$ gives $|\psi'|^2 = \psi'^* e^{-i\alpha} e^{i\alpha} \psi = |\psi|^2$.

1.2.3 Orthogonal and special orthogonal groups

If we restrict the unitary matrices to be real, we get the orthogonal groups.

Definition: The **orthogonal group** $O(n)$ is the group of real $n \times n$ orthogonal matrices O , *i.e.* matrices where $O^T O = I$. If we additionally require that $\det(O) = 1$ the matrices form the **special orthogonal group** $SO(n)$.

It follows from the definition of the orthogonal group that the determinant of the members is either 1 or -1 , thus the special orthogonal group is simply one half of the members. For $\mathbf{x} \in \mathbb{R}^n$ the orthogonal group has the same property as the unitary group of leaving the length of vectors invariant.

Matrices in the $O(n)$ and $SO(n)$ groups have $n(n-1)/2$ independent parameters since an $n \times n$ matrix with real entries has n^2 elements, and there are $n(n+1)/2$ equations to be satisfied by the orthogonality condition.⁵

The special orthogonal groups $SO(2)$ and $SO(3)$ are much used because their elements represent rotations in two and three dimensions, respectively, while $SO(n)$ extends this to higher dimensions and represents the symmetries of a sphere in n dimensions. To see this we can start from the fact that rotations, by definition, conserve angles and distances (and orientation). This means that the original set of orthogonal axis – or orthogonal basis vectors if you wish – must transform into another orthogonal set of axis under the rotation. The matrix performing the rotation must then be orthogonal, and thus the collection of rotations must be $O(n)$. If we additionally require that orientation is preserved, this removes the matrices with negative determinant, leaving the $SO(n)$ group.

⁵Only the upper triangular part of $O^T O$ has independent equations since $O^T O$ is a symmetric matrix, $(O^T O)^T = O^T (O^T)^T = O^T O$.

$SO(2)$

Given that $SO(2)$ has only one parameter, we can write a general group member R as parameterised by θ

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

As expected, we can recognise this as the matrix of rotations of an angle θ around a point in the plane, and it represents the (orientation preserving) symmetries of a circle. Some tinkering with this representation will show that $SO(2)$ is in fact an abelian group, despite the matrix definition. From a physical viewpoint this should be expected: the order of rotations in the plane should not matter.

It is interesting to observe that the elements of $SO(2)$ rotate points in the plane, while the elements of $U(1)$ rotate complex numbers, which can be represented by points in the plane. Indeed, a one-to-one correspondence can be found between the members of the two groups so that the groups are indeed the same, or, as we say, isomorphic, $SO(2) \cong U(1)$.

 $SO(3)$

This group has three free parameters. Already at this point writing down the explicit form of a general group member is not very enlightening. There are also a number of different conventions in use, so proper care is advised when using results from the literature. In terms of a general rotation in three dimensions this can either be viewed as rotation angles around three fixed axis, or as the fixing of a rotation axis by two angles, with a third rotation angle around that axis.

One particular explicit form, where the angles correspond to the three **Euler angles** of rotation in three dimensions, α , β and γ , is

$$R(\alpha, \beta, \gamma) = \begin{bmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \\ \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \\ -\sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \end{bmatrix},$$

where $0 \leq \alpha, \gamma < 2\pi$ and $0 \leq \beta \leq \pi$. These rotations do not commute, so the group is non-abelian.

We have seen above that $SU(2)$ has three free parameters, just the same as $SO(3)$. You may at this point guess that $SO(3)$ is isomorphic to $SU(2)$. That would be a very good guess, however, it would also be wrong. We will return to this later, but this is one of those things in mathematics that turn out to be disappointingly only almost true.

1.3 Group properties

1.3.1 Subgroups

We now extend our vocabulary for groups by defining the **subgroup** of a group G .

Definition: A subset $H \subset G$ is a **subgroup** if and only if:^a

- i) $h_i \circ h_j \in H$ for $\forall h_i, h_j \in H$, (closure)
- ii) $h_i^{-1} \in H$ for $\forall h_i \in H$. (inverse)

H is a **proper** subgroup if and only if $H \neq G$ and $H \neq \{e\}$.

^aAn alternative, equivalent, and more compact way of writing these two requirements is the single requirement $h_i \circ h_j^{-1} \in H$ for $\forall h_i, h_j \in H$. This is often utilised in proofs.

We have already seen some examples of subgroups: the $SU(n)$ groups are subgroups of $U(n)$, and the $SO(n)$ groups are subgroups of the $O(n)$ groups. This can easily be shown using the properties of determinants.

There is a very important class of subgroup called the **normal** subgroup. The importance will become clear in a moment.

Definition: A subgroup H is a **normal** (invariant) subgroup, if and only if the **conjugation** of any element by any $g \in G$ is in H .^a meaning

$$ghg^{-1} \in H \text{ for } \forall h \in H.$$

A **simple** group G has no proper normal subgroup. A **semi-simple** group G has no proper abelian normal subgroup.

^aAnother, pretty, but slightly abusive, way of writing the definition of a normal group is to say that $gHg^{-1} = H$. This implies (correctly), that the image of H under the conjugation operation is guaranteed to be the whole of H .

We can for example show that for $n > 1$, $SU(n)$ is a normal subgroup of $U(n)$ (see exercises).

1.3.2 Quotient groups

The normal subgroup can be seen as a factor in the original group that can be divided out to form a simpler group that only retains the structure that was not in the normal group. To be more precise we need the concept of **cosets**.

Definition: A **left coset** of a subgroup $H \subset G$ with respect to $g \in G$ is the set of members $\{gh|h \in H\}$, and a **right coset** of the subgroup is the set $\{hg|h \in H\}$. These are sometimes written gH and Hg , respectively.

For normal subgroups H it can be shown that the sets of left and right cosets coincide and form a group. This is called the **quotient** or **coset group** and denoted G/H .⁶ This has as its members all the *distinct sets* $\{gh|h \in H\}$, that can be generated by a $g \in G$, and has the binary operation $*$ with $\{gh|h \in H\} * \{g'h|h \in H\} = \{(g \circ g')h|h \in H\}$. To simplify notation this can be written $gH * g'H = (g \circ g')H$.

Let us briefly discuss an example of a quotient group. We already know that $SU(n)$ is a normal subgroup to $U(n)$. This means that $U(n)/SU(n)$ is a group. What sort of group is this? Notice that two matrices U_i and U_j live in the same coset of $SU(n)$ if and only if

⁶Sometimes also called the **factor group**. The notation is pronounced “G mod N”, where “mod” is short for modulo.

$\det(U_i U_j^{-1}) = 1$, which is true if and only if $\det(U_i) = \det(U_j)$. In other words, each coset constructed from $SU(n)$ is simply the set of all matrices with a given determinant, which we saw for the $U(n)$ group can be any unit complex number. Observe that this means that many of the cosets are the same set. Here, all cosets generated by members in $U(n)$ with the same determinant is the same coset. When these cosets act on each other with the group operation of $U(n)/SU(n)$ they form new cosets of matrices with a determinant that is the product of their individual determinants, *i.e.* with $U, U' \in U(n)$ we have the product of quotient group member $\{US|S \in SU(n)\} * \{U'S|S \in SU(n)\} = \{UU'S|S \in SU(n)\}$. Thus, the group behaves exactly as $U(1)$, and is in fact isomorphic to it.

1.3.3 Product groups

Now that we have introduced group division, we also need to introduce products of groups.

Definition: The **direct product** of groups G and H , $G \times H$, is defined as the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \circ (g_j, h_j) = (g_i \circ g_j, h_i \circ h_j)$. $G \times H$ is then a group and G and H can be shown to be normal subgroups of $G \times H$.

We should note there that the subgroups are strictly $G \times \{e_H\}$ and $\{e_G\} \times H$, but these are isomorphic to G and H .

Because it has at least one important guest star appearance in this text we also need the definition of the **semi-direct product**.

Definition: The **semi-direct product** of groups G and H , $G \rtimes H$, where G is also a mapping $G : H \rightarrow H$, is defined by the ordered pairs (g, h) where $g \in G$ and $h \in H$, with component-wise operation $(g_i, h_i) \circ (g_j, h_j) = (g_i \circ g_j, h_i \circ g_i(h_j))$. Here H is not a normal subgroup of $G \rtimes H$, but G is.

Note how the semi-direct product is not symmetric between the factors.

The famous Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ is an example of a direct product. Direct products are “trivial” structures because there is no “interaction” between the subgroups, the elements of each group act only on elements of the same group. This is not true for semi-direct products.

1.3.4 Isomorphic groups

We have already talked about how two groups are the same if they have a correspondence between their members and the same results for the multiplication operation, and we have called this an isometry. Let us now try to put this notion of when two groups are the same into a more formal language.

Definition: Two groups G and H are **homomorphic** if there exists a map between the elements of the groups $\rho : G \rightarrow H$, such that for $\forall g, g' \in G$, $\rho(g \circ g') = \rho(g) \circ \rho(g')$.

For homomorphic groups we say that the mapping conserves the structure of the group, or in other words, all the rules for the group operation/multiplication. This leads to our notion of group equality, namely **isomorphic** groups:

Definition: Two groups G and H are **isomorphic**, written $G \cong H$, if they are homomorphic and the relevant mapping is one-to-one.

The one-to-one mapping ensures that there is a one-to-one correspondence between the elements of the group, so that isomorphic groups effectively contain both the same members and have the same multiplication operation.

For matrix groups, a good way of checking the plausibility of isomorphism is to count the number of free parameters. The difference of the parameters for the two factors should be equal to the number of parameters for the quotient group. In our example $U(n)/SU(n)$, $U(n)$ has n^2 , while $SU(n)$ has $n^2 - 1$, and this gives $n^2 - (n^2 - 1) = 1$ parameters for the quotient group, which matches the one parameter of $U(1)$.

1.4 Representations

In some contrast to the treatment in most introductory group theory texts in mathematics, physicists are mostly interested in the properties of groups G where the elements of G act to transform some elements of a vector space $v \in V$, $g(v) = v' \in V$.⁷ Here, the members of V can for example be the state of a system, say a wave-function in quantum mechanics or a field in quantum field theory. To be useful in physics, we would like that the result of the group operation $g_i \circ g_j$ acts as $(g_i \circ g_j)(v) = g_i(g_j(v))$ and the group identity acts as $e(v) = v$.

We begin with the abstract definition of a representation that ensures these properties.

Definition: A **representation** of a group G on a vector space V is a map $\rho : G \rightarrow GL(V, K)$, where $GL(V, K)$ is the **general linear group** on V , *i.e.* the invertible matrices of the field K of V .^a such that for $\forall g_i, g_j \in G$,

$$\rho(g_i \circ g_j) = \rho(g_i)\rho(g_j). \text{ (homomorphism)}$$

If this map is also isomorphic, we say that the representation is **faithful**.

^aTechnically, we can only be sure that we can write $GL(V, K)$ as matrices as long as V is a finite dimensional vector space. However, we shall do our best not to ass around with infinite dimensional representations.

From a physicist point of view, the underlying point here is that (the members of) our groups will be used on quantum mechanical states, or fields in field theory, which can be just complex numbers (functions) or multi-component vectors of such. They are thus members of a vector space, and the definition of representations force the transformation properties of the group to be written in terms of matrices. Furthermore, the mapping from the group, or, if you like, the concrete way of writing the abstract group elements, must be homomorphic (structure preserving), meaning that if we can write a group element as the product of two others, the matrix for that element must be the product of the two matrices for the individual group elements it can be written in terms of.

You may by now have realized that the matrix groups defined in Sec. 1.2 have the property that they are defined in terms of one of their representations. These are called the **fundamental** or **defining representations**. However, we will also have use for other representations,

⁷This part of group theory is called *representation theory*.

e.g. the **adjoint representation** which we will introduce below.

Let us now take a few examples that connect to our definition. We saw earlier that for $U(1)$ the group members can be written as the complex numbers on the unit circle $e^{i\alpha}$, which can be used as phase transformations on wavefunctions $\psi(x)$ that form a one dimensional vector space over the complex numbers. For $SU(2)$ we can show that the group elements can be written in the fundamental representation as $e^{i\alpha_i\sigma_i}$, with σ_i being the **Pauli matrices**.

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (1.1)$$

In the Standard Model this is applied to weak doublets of fields, *e.g.* the electron–neutrino doublet $\psi = (\nu_e, e)$ that form a two-dimensional vector space, as the $SU(2)_L$ gauge transformation^[8]

For later use we need to know when two representations are equivalent.

Definition: Two representations ρ and ρ' of G on V and V' are **equivalent** if and only if $\exists A : V \rightarrow V'$, that is one-to-one, such that for $\forall g \in G$, $A\rho(g)A^{-1} = \rho'(g)$.

The building blocks of representations are so-called **irreducible representations**, also called **irreps**. These are the essential ingredients in representation theory, and are defined as follows:

Definition: An **irreducible representation** ρ is a representation where there is *no* proper subspace $W \subset V$ that is closed under the group, *i.e.* there is no $W \subset V$ such that for $\forall w \in W$, $\forall g \in G$ we have $\rho(g)w \in W$.^[a]

^aIn other words, we can not split the matrix representation of G in two parts that do not “mix”.

Let us take an example to try to clear up what a *reducible* representation means in contrast to an irreducible. Assume the representation $\rho(g)$ for $g \in G$ acts on a vector space V as matrices. If these matrices $\rho(g)$ can all be decomposed into $\rho_1(g)$, $\rho_2(g)$ and $\rho_{12}(g)$ such that for $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in V$

$$\rho(g)\mathbf{v} = \begin{bmatrix} \rho_1(g) & \rho_{12}(g) \\ 0 & \rho_2(g) \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix},$$

then ρ is **reducible**. The subspace W of V spanned by $\mathbf{v}_1$ violates the irreducibility condition above.

If we also have $\rho_{12}(g) = 0$ we say that the representation is **completely reducible**. It can be shown that in most cases a reducible representation is also completely reducible. In fact, representations for which this is not true tend to be mathematical curiosities. As a result, there is a tendency in physics to use the term “reducible” where we should use the term “completely reducible”. In the case of a completely reducible representation we can split the vector space V into two vector spaces $V = W_1 \oplus W_2$, where $\mathbf{v}_1 \in W_1$ and $\mathbf{v}_2 \in W_2$, and

⁸We will return to why we want to write the group as this particular exponential in Sec. 1.7 however, a quick count of the number of free parameters in $SU(2)$ should convince us that we need three real numbers to parametrise the group elements, here the α_i : each matrix consists of four complex numbers, or eight real, the unitarity requirement removes four free parameters, while the requirement on the determinant one more. Since the three Pauli matrices are linearly independent, $\alpha_i\sigma_i$ should then span $SU(2)$ and the exponentiation does not change that.

define a representation of G on each of them using ρ_1 and ρ_2 , which in turn could either be reduced more, or would be irreducible.

We end this section with an important theorem that helps us decide whether a representation is irreducible, and ultimately gives a property identifying the representation. As many important theorems, it is called a lemma.

Theorem: (Schur's Lemma)

If we have an irreducible representation ρ of a group G , all the matrices A that commute with $\rho(g)$ for $\forall g \in G$ are proportional to the identity, $A = \lambda I$. Here the λ are constants that label the representation.

1.5 Lie groups

In physics we are particularly interested in a special type of group, the **Lie group**, a class of continuous groups that we can parametrise and which are the basic tool we use to describe continuous symmetries. In order to define Lie groups we will need to use the technical term (smooth) manifold, meaning a mathematical object (formally a topological space) that locally⁹ can be parametrised as a function of $\mathbb{R}^n$ or $\mathbb{C}^n$. We will describe a Lie group G in terms of a parameterisation of the members $g(\vec{a}) \in G$, where $\vec{a} \in \mathbb{R}^n$ (or $\mathbb{C}^n$). Additionally, in order to describe continuous symmetries these parameterisations need to be *smooth*, also in the technical sense of smooth, which means infinitely differentiable.

Definition: A **Lie group** G is a finite-dimensional **smooth manifold** where group multiplication and inversion are smooth functions, meaning that given elements $g(\mathbf{a}), g'(\mathbf{a}') \in G$, $g(\mathbf{a}') \circ g'(\mathbf{a}') = g''(\mathbf{b})$ where $\mathbf{b}(\mathbf{a}, \mathbf{a}')$ is a smooth function of $\vec{a}$ and $\vec{a}'$, and $g^{-1}(\mathbf{a}) = g'(\mathbf{a}')$ where $\mathbf{a}'(\mathbf{a})$ is a smooth function of $\vec{a}$.

In terms of the situation we are often interested in, a Lie group G acting on a vector space V through a representation, it can be shown (for finite-dimensional representations) that we can write the map of the representation $G \times V \rightarrow V$ for $\mathbf{x} \in V$ as $x_i \rightarrow x'_i = f_i(\vec{x}, \vec{a})$ where f_i is analytic¹⁰ in x_i and a_i . Additionally f_i has an inverse.

We immediately see that the translation group $T(1)$ with the parameterisation $g(a) = e^{a \frac{d}{dx}}$ is a Lie group since $g(a) \cdot g(a') = g(a + a')$ and $a + a'$ is an analytic function of a and a' . Here we can write the action of the group on the vector space $\mathbb{R}^1$ as $f(x, a) = x + a$. The $SU(n)$ groups are also Lie groups as they have a fundamental representation $e^{i\alpha\lambda}$ where λ is a set of $n \times n$ -matrices, and $f_i(\mathbf{x}, \alpha) = [e^{i\alpha\lambda}\mathbf{x}]_i$.

By the analyticity we can always construct the parametrization so that $g(0) = e$ or $x_i =$

⁹This insistence on local means that the parameterisation is not necessarily the same for the whole group.

¹⁰Meaning infinitely differentiable and in possession of a convergent Taylor expansion. As a result analytic functions (on $\mathbb{R}$) are smooth, but the reverse does not hold.