Interaction theory – charged particles

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Excitation / ionization

• Incoming charged particle interacts with atom / molecule:

- An ion pair is created

Elastic collision 1

• Interaction between two particles where kinetic energy is preserved:

- Classical mechanics:

\[
\begin{align*}
E_i &= \frac{1}{2} m_v \nu_i^2 = \frac{1}{2} m_{v_1} \nu_1^2 + \frac{1}{2} m_{v_2} \nu_2^2 \\
E &= m_v \nu_1 \cos \theta + m_{v_2} \nu_2 \cos \chi \\
0 &= m_v \nu_1 \sin \theta - m_{v_2} \nu_2 \sin \chi
\end{align*}
\]

Elastic collision 2

\[
\begin{align*}
\Rightarrow \nu_i &= \frac{2 m_v \nu \cos \chi}{m_1 + m_2}, \quad \nu = \nu_1 = \sqrt{\frac{1 - \frac{4 m_v \nu \cos^2 \chi}{(m_1 + m_2)^2}}}{m_1 - \cos 2 \chi} \\
\tan \theta &= \frac{\sin 2 \chi}{m_1 - \cos 2 \chi} \\
\text{η} &= \frac{1}{2} m_{\nu_2} \nu_2^2 = \frac{4 m_v m_2}{(m_1 + m_2)^2} T_0
\end{align*}
\]
\begin{itemize}
  \item Proton-electron collision:
    \[ \theta_{\text{max}} = 0.03^\circ, \quad E_{\text{max}} = 0.2 \% \]
  \item Electron-electron (or e.g. proton-proton) coll.:
    \[ \theta_{\text{max}} = 90^\circ, \quad E_{\text{max}} = 100 \% \]
\end{itemize}

Elastic collision 3

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 &gt; m_2 )</td>
<td>[ 0 \leq \chi \leq \pi/2 ]</td>
</tr>
<tr>
<td>( m_1 = m_2 )</td>
<td>[ 0 \leq \chi \leq \pi/2 ]</td>
</tr>
<tr>
<td>( m_1 &lt; m_2 )</td>
<td>[ 0 \leq \chi \leq \pi/2 ]</td>
</tr>
</tbody>
</table>

\[
E_{\text{max}} = \frac{4m_1T_0}{m_1T_0}
\]

Elastic collision – cross section 1

\[
\tilde{F} = \frac{zZe^2}{4\pi\epsilon_0b^2} \tilde{u}
\]

Force exerted on particle 2:
\[
F_x = F\sin \eta, \quad F_y = F\cos \eta
\]

Elastic collision – cross section 2

Momentum of particle 2:
\[
d\tilde{p}_x = \tilde{F}dt
\]
\[
\frac{dx}{dt} = v, \quad \tan \eta = \frac{x}{b}
\]
\[
\Rightarrow \frac{d}{d\eta} \tan \eta = \frac{1}{\cos^2 \eta} = \frac{dx}{b d\eta} \quad \Rightarrow \quad dt = \frac{bd\eta}{v \cos^2 \eta}
\]

Total momentum transfer in interaction:
\[
\tilde{p}_x = \int \cos \eta \frac{bd\eta}{\cos^2 \eta} = \frac{zZe^2}{4\pi\epsilon_0v} \frac{b^2}{2} \int \cos \eta \frac{1}{\cos \eta} \quad , \quad r = \frac{b}{\cos \eta}
\]
\[
\Rightarrow \tilde{p}_x = \frac{zZe^2}{4\pi\epsilon_0v} \frac{b^2}{2} \int \cos \eta d\eta = \frac{2zZe^2}{4\pi\epsilon_0v} \frac{b^2}{2}
\]

Elastic collision – cross section 3

Energy transfer:
\[
E = \frac{p_x^2}{2m_2} = \frac{2}{m_2} \left( \frac{zZe^2}{4\pi\epsilon_0v} \right)^2 \frac{1}{E}
\]

Cross section:
\[
\sigma = \pi b^2 \quad \Rightarrow \quad d\sigma = 2\pi b dbd. \text{ Thus:}
\]
\[
b^2 = \frac{2}{m_2} \left( \frac{zZe^2}{4\pi\epsilon_0v} \right)^2 \frac{1}{E} \quad \Rightarrow \quad \sigma = 2\pi \frac{2}{m_2} \left( \frac{zZe^2}{4\pi\epsilon_0v} \right)^2 \frac{1}{E} dE
\]
\[
\frac{r_y}{r_x} = \frac{\frac{b^2}{4\pi\epsilon_0m_1c^2}}{1}
\]
\[
\Rightarrow \frac{d\sigma}{dE} = 2\pi \left( \frac{zZe^2}{m_1v} \right)^2 \frac{1}{E^2} \quad \Rightarrow \quad \sigma = 2\pi \left( \frac{zZe^2}{m_1v} \right)^2 \frac{1}{E^2}
\]
Elastic collision – cross section 4

- Consider $z=1$ and $m_1 = m_e << m_2$

$$m_1 << m_2 \quad \Rightarrow \quad E = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_2 \left( \frac{2m_1 v \cos \chi}{m_2} \right)^2 = \frac{m_1}{m_2} m_e v^2 \cos^2 \chi$$

$$\tan \theta = \frac{\sin \chi}{\cos \chi} = -\tan 2\chi \quad \Rightarrow \quad \chi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\Rightarrow \quad \frac{d\sigma}{d\theta} = \frac{d\sigma}{dE \, d\theta} \quad \Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{2\pi \sin \theta \, d\theta} = \frac{Z^2 e^2 m_e c^2}{4} \frac{1}{\beta^2 \sin^4(\theta/2)}$$

**Stopping power**

- $S = \frac{dT}{dx}$; expected energy loss per unit length

$$\tau_s = T_s \cdot n_v$$

$$dT = \left( E_n, d\sigma \right) = n_v \int dx \frac{d\sigma}{dE} dE$$

$$\frac{dT}{d\rho dx} = \frac{S}{\rho} = \left( \frac{N_Z}{A} \right)^b \int d\sigma \frac{d\sigma}{dE} dE$$

- $S = \frac{dT}{\rho dx}$: mass stopping power

**Impact parameter**

- Charged particles: Coulomb interactions
- Most important: interactions with electrons
- Impact parameter $b$:

**Soft collisions 1**

- $b >> a$: incoming particle passes atom at long distance
- Weak forces, small energy transfers to the atom
- Inelastic collisions: Predominantly excitations, some ionizations
- Energy transfer range from "$E_{min}$" to "$H$"
- Hans Bethe: Quantum mechanical considerations
- Theory for heavy charged particles in the following
Soft collisions 2

\[
S_{\text{soft}} = \frac{dS_{\text{soft}}}{\rho} = \frac{N_e Z 2\pi\varepsilon_0 m_e c^2 z}{\beta^2} \left[ \frac{2m_e c^2 \beta}{\ln \left( \frac{H}{1-\beta^2} \right) - \beta^2} \right]
\]

- \( r_0 \): classical electron radius = \( e^2/4\pi\varepsilon_0 m_e c^2 \)
- \( I \): mean excitation potential
- \( \beta = v/c \)
- \( z \): charge of incoming particle
- \( \rho \): density of medium
- \( N_e Z/A \): numbers of electron per gram
- \( H \): maximum energy transferred by soft collisions

Soft collisions 2

- Quantum mechanics (atomic structure) is reflected in the mean excitation potential

Hard collisions 1

- \( b \sim a \): charged particle pass 'through' atom
- Large (but few) energy transfers
- Energy transfers from \( H \) to \( E_{\max} \)
- May be considered as an elastic collision between free particles (binding energy is negligible)

\[
S_{\text{hard}} = \frac{dS_{\text{hard}}}{\rho} = \frac{N_e Z 2\pi\varepsilon_0 m_e c^2 z^2}{\beta^2} \left[ \ln \left( \frac{E_{\max}}{H} \right) - \beta^2 \right]
\]

Collision stopping power

- For inelastic collisions, the total cross section is thus:

\[
S = S_{\text{soft}} + S_{\text{hard}} = 4\pi\varepsilon_0 m_e c^2 \left( \frac{N_e Z}{A} \right) \left[ \frac{z^2}{\beta^2} \ln \left( \frac{2m_e c^2 \beta}{(1-\beta^2)^2} \right) - \beta^2 \right]
\]

- Important: Increases with \( z^2 \), decreases with \( \beta^2 \) and \( I \), not dependent on particle mass
$S_c/\rho$, different substances

- $I$ and electron density ($ZN_A/A$) give differences

$S_c/\rho$, electrons and positrons

- Electron-electron scattering is more complicated; scattering between two identical particles
- $S_{c,\text{hard}}/\rho$ (el-el) is described by the Möller cross section
- $S_{c,\text{hard}}/\rho$ (pos-el) is described by the Bhabha c.s.
- $S_{c,\text{soft}}/\rho$ was given by Bethe, as for heavy particles
- Characteristics similar to that for heavy charged particles

Shell correction

- Derivation of $S_c$ assumes $v \gg v_{\text{atomic electrons}}$
- When $v \sim v_{\text{atomic electrons}}$, no ionizations
- Most important for K-shell electrons
- Shell correction $C/Z$ takes this into account, and thus reduces $S_c/\rho$
- $C/Z$ depends on particle energy and medium

Density correction

- Charged particles polarizes medium which is being traversed

\[
\begin{align*}
&\mathbf{E}_\text{part} = \mathbf{E}_\text{part} + \mathbf{E}_\text{ion} - \mathbf{E}_\text{pol} \\
&\text{Charged (+e) particle}
\end{align*}
\]

- Weaker interactions with remote atoms due to reduction in electromagnetic field strenght
- Polarization increases with energy and density
- Most important for electrons and positrons
Density correction

- Density correction $\delta$ reduces $S_c/\rho$ for liquids and solids
- $S_c/\rho$ (water vapor) > $S_c/\rho$ (water)

\[
\frac{S_c}{\rho} (\text{water vapor}) > \frac{S_c}{\rho} (\text{water})
\]

Linear Energy Transfer 1

- LET$_\Delta$ is denoted the restricted stopping power
- $dT/dx$: mean energy loss per unit length – but how much is deposited ‘locally’?

\[
\frac{dE}{dx} = \frac{E_{\text{min}}}{E_{\text{max}}} \text{ or } E_{\text{min}} < E < E_{\text{max}}
\]

- For $\Delta = E_{\text{max}}$, we have $L_\infty = S_c$; unrestricted LET
- LET$_\Delta$ is often given in [keV/µm]
- 30 MeV protons in water: LET$_{100\text{ ev}} / L_\infty = 0.53$

Linear Energy Transfer 2

- Energy loss (soft + hard) per unit length for $E_{\text{min}} < E < \Delta$:

\[
L_\Delta = \left( \frac{dT}{dx} \right)_\Delta = \rho \left( \frac{N_A Z}{A} \right) \int_{E_{\text{min}}}^{\Delta} dE \frac{dE}{dx}
\]

\[
= \rho 2\pi\epsilon_0 \frac{e^2}{c^3} \left( \frac{N_A Z}{A} \right) \left( \frac{Z}{\beta} \right)^2 \left[ \ln \left( \frac{2m_e c^2 \beta^2 \Delta}{(1-\beta^2)^2} \right) - 2\beta^2 \right]
\]

- For $\Delta = E_{\text{max}}$, we have $L_\infty = S_c$; unrestricted LET
- LET$_\Delta$ is often given in [keV/µm]
- 30 MeV protons in water: LET$_{100\text{ ev}} / L_\infty = 0.53$

Brehmsstrahlung 1

- Photon may be emitted from charged particle accelerated in the field from an electron or nucleus

\[
P = \frac{(ze)^2 a^2}{6\pi\epsilon_0 c^2}
\]

- Larmor’s formula (classical electromagnetism) for radiated effect from accelerated charged particle:
Brehmsstrahlung 2

- For particle accelerated in nuclear field:
  \[ F = ma = \frac{zZe^2}{4\pi\varepsilon_0 r^2} \Rightarrow a = \frac{zZe^2}{4\pi\varepsilon_0 mr^2} \]
  \[ \Rightarrow P \propto \left( \frac{Z}{m} \right)^2 \]

- Comparison of protons and electrons:
  \[ \frac{P_p}{P_e} \approx \frac{1}{1836^2} \]

- Brehmsstrahlung not important for heavy charged particles

Brehmsstrahlung 3

- Energy loss by brehmsstrahlung is called *radiative loss*
- Maximum energy loss is the total kinetic energy \( T \)
- Radiative loss per unit length: *radiative stopping power*:
  \[ \left( \frac{S}{\rho} \right)_p = \left( \frac{dT}{dx} \right) = \alpha \beta Z^2 (T + m_c c^2) \overline{B}(T, Z) \]
  \[ \overline{B}(T, Z) \text{ weakly dependent on } T \text{ and } Z \]
- Brehmsstrahlung increases with energy and atomic number

Total stopping power, electrons

\[ \left( \frac{dT}{dx} \right)_{\text{tot}} = \left( \frac{dT}{dx} \right)_e + \left( \frac{dT}{dx} \right)_\gamma \]

Radiation yield

\[ Y(T) = \frac{S}{S} \frac{TZ}{n} \approx n=750 \text{ MeV} \]
S/r, protons and electrons

- High energy electrons ($v > c/n$) polarizes medium (e.g. water) and blueish light (+ UV) is emitted
- Low energy loss

Other interactions

- **Nuclear interactions**: Inelastic process where charged particle (e.g. proton) excites nucleus →
  - Scattering of charged particle
  - Emission of neutron, photon, or $\alpha$-particle ($\text{^4He}$)
- Not important below ~10 MeV (protons)

- **Positron annihilation**: Positron interacts with electron → a pair of photons with energy $\geq 2 \times 0.511$ MeV is created. Photons are emitted in opposite directions.
- Probability decreases as $\sim 1/v$

Charged particle interactions, summary
Range 1
- *The range* $\mathcal{R}$ of a charged particle in matter is the (expectation value) of its total pathlength $p$
- *The projected range* $<t>$ is the (expectation value) of the largest depth $t_f$ a charged particle can reach along its incident direction
- Electrons: $<t> < \mathcal{R}$
- Heavy charged particles: $<t> \approx \mathcal{R}$

CSDA-range
- The range may be approximated by $\mathcal{R}_{\text{CSDA}}$ (continuous slowing down approximation)
- Energy loss per unit length $dT/dx$ – gives implicitly a measure of the range:

$$\mathcal{R}_{\text{CSDA}} = \frac{T_0}{d} \int \frac{dT}{dx} \, dT$$

Range 3
- The range is often given multiplied by the density:
  $$\mathcal{R}_{\text{CSDA}} = \int \left( \frac{dT}{dx} \right)^{-1} \, dT$$
- Unit thus becomes [cm] [g/cm$^3$] = [g/cm$^2$]
- Range of charged particle depends on:
  - Charge and kinetic energy
  - Density, electron density and mean excitation potential of absorber

Range 4
- The range of charged particles depends on:
  - Charge and kinetic energy
  - Density, electron density and mean excitation potential of absorber

- Unit thus becomes [cm] [g/cm$^3$] = [g/cm$^2$]
Multiple scattering and straggling

- In a beam of charged particles, one has:
  - Variations in energy deposition (straggling)
  - Variations in angular scattering

→ The beam, where all particles originally had the same velocity, will be smeared out as the particles traverse matter

Straggling

![Straggling Diagram]

Range issues

![Range Issues Diagram]

Energy deposition, protons

![Energy Deposition Diagram]
Energy deposition, electrons

Monte Carlo simulations
• Monte Carlo simulations of the track of an electron (0.5 keV) and an α-particle (4 MeV) in water
• Note: $e^-$ is most scattered, $\alpha$ has the highest $dT/dx$

Hadron therapy
• Heavy charged particles may be used for radiation therapy – conforms better to the target than photons or electrons

Web pages
• For stopping powers:
  http://www.nist.gov/pml/data/star
• For attenuation coefficients:
  http://www.nist.gov/pml/data/xraycoef