Motion of charged particles in EM fields

We consider plasmas in which no collisions take place. The basic equation of motion of a charged particle in an electromagnetic field is then

$$m\vec{a} = m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{1}$$

0.1 Energy considerations

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{d}{dt}W = q\vec{E}\cdot\vec{v} \tag{2}$$

Therefore:

☞ No spatial change in the magnetic field results in a change of the total kinetic energy.

☞ Electric fields perpendicular to the direction of motion do not change the total kinetic energy.

☞ Only electric fields aligned with the direction of motion can change the total kinetic energy.

0.2 Motion in steady magnetic field, $\vec{E} = 0$

Assume $\vec{E} = 0$ and $\vec{B} = B_0\vec{e}_z$. Then from (1)

$$\frac{dv_x}{dt} = \frac{q}{m}v_y B_0 \tag{3}$$

$$\frac{dv_y}{dt} = -\frac{q}{m}v_x B_0 \tag{4}$$

$$\frac{dv_z}{dt} = 0 \rightsquigarrow v_z \text{ const.} \tag{5}$$

Differentiate (3) and substitute (4):

$$\frac{d^2v_x}{dt^2} = \frac{qB_0}{m}\frac{dv_y}{dt} = -\left(\frac{qB_0}{m}\right)^2 v_x \tag{6}$$

This describes a simple harmonic oscillator with a frequency of $\omega_g = qB_0/m$ with the solution:

$$v_x(t) = A\cos(\omega_g t + \phi) \tag{7}$$

$$v_y(t) = -A\sin(\omega_g t + \phi) \tag{8}$$

where

$$v_x^2 + v_y^2 = A^2 = v_\perp^2. \tag{9}$$

To get the actual movement of the particle, integrate the solution:

$$x(t) = \int v_x(\tau)d\tau = x_0 + \frac{v_\perp}{\omega_g}\sin(\omega_g t + \phi) \tag{10}$$

$$y(t) = \int v_y(\tau)d\tau = y_0 + \frac{v_\perp}{\omega_g}\cos(\omega_g t + \phi) \tag{11}$$

Here, x_0 and y_0 are constants of integration and ϕ is the initial phase. The particle moves in a circle, perpendicular to the background magnetic field. It "gyrates". The radius of the gyromotion, the gyroradius, is

$$r_g = \frac{v_\perp}{\omega_g} = \frac{mv_\perp}{qB_0} \tag{12}$$

where ω_g is the gyrofrequency:

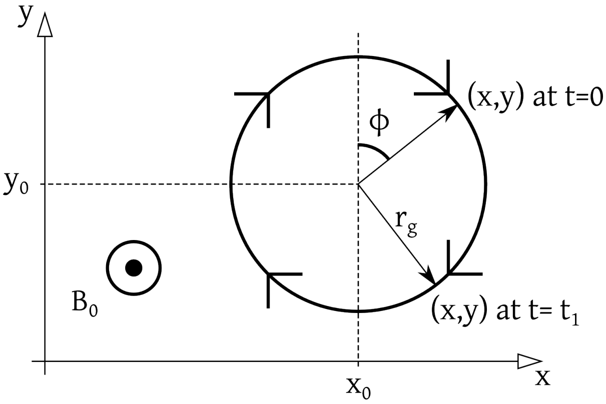

Figure 1: Sketch of the gyromotion of a positively charged particle.

Because of the dependence on q, the direction of the gyromotion of positively charged particles (ions) is opposite that of negatively charged particles (electrons). When considering the gyromotion of electrons and ions, for the same perpendicular energy, the gyrofrequency of electrons is much higher ($m_e \ll m_i$) while the gyroradius is much larger. (5) shows that v_z, i.e., the velocity along the field is constant. Therefore, for $v_z \neq 0$, the particle trajectory is helical around the magnetic field line, and the center of the gyromotion is called the "guiding center".

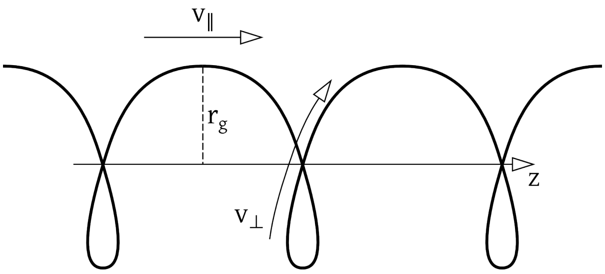

Figure 2: Sketch of the gyromotion for $v_{\|} \neq 0$.

0.3 Pitch angle

The pitch angle α is defined by the ratio of $v_\perp$ and $v_\|$:

$$\alpha = \arctan \frac{v_\perp}{v_\|} \tag{13}$$

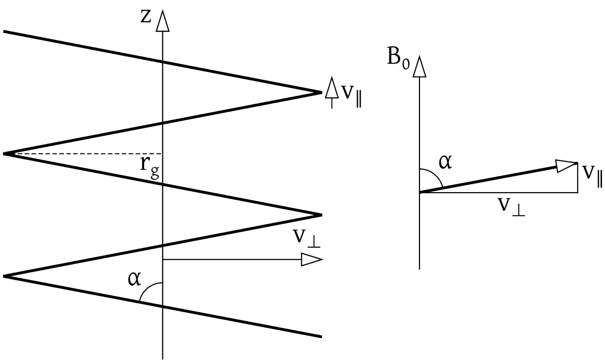

Figure 3: Pitch angle for the case for $v_\| < v_\perp$.

0.4 Magnetic moment

Total kinetic energy:

$$W = W_\perp + W_\| \tag{14}$$

For $\vec{E} = 0$, $dW/dt = dW_\perp/dt + dW_\|/dt = 0$. Define magnetic moment as $\mu = W_\perp/B$, then

$$\frac{dW_\perp}{dt} = B\frac{d\mu}{dt} + \mu\frac{dB}{dt}. \tag{15}$$

Also

$$\frac{dW_\|}{dt} = -\mu\frac{dB}{dt}, \tag{16}$$

such that

$$\frac{dW_\perp}{dt} + \frac{dW_\|}{dt} = B\frac{d\mu}{dt} = 0 \tag{17}$$

and hence $d\mu/dt = 0$, meaning that the magnetic moment μ is conserved as the particle moves through spatially varying magnetic fields.

0.5 Magnetic mirror

Consider the following situation where the magnetic field converges:

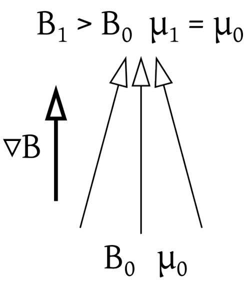

Figure 4: Converging magnetic field.

Because

$$\mu = \frac{W_\perp}{B} = \frac{mv_\perp^2}{2B} = \text{const.} \tag{18}$$

as B increases $v_\perp$ must also increase. However, the total kinetic energy must not change, such that $v_\|$ must decrease. That increases the pitch angle (see (13)), meaning that the movement along the field line slows down. Once the field-aligned movement stops, there is still a force $\vec{F} = -\mu\nabla B$

on the gyrating particle, pushing it back out of the region of higher B.
Define total speed v as $v^2 = v_\parallel^2 + v_\perp^2$, then $v_\perp = v \sin\alpha$ and $\mu = mv^2 \sin^2\alpha/2B$. Then because $\mu_0 = \mu_1$:

$$\frac{\sin^2\alpha_0}{\sin^2\alpha_1} = \frac{B_0}{B_1} \tag{19}$$

0.6 Trapped particles in a dipole field

Earth's magnetic field is essentially a dipole field where $B(r) \propto 1/r^3$. As particles move along field lines toward the magnetic poles, they experience stronger magnetic fields and are eventually mirrored. As they gyrate they are said to "bounce" between hemispheres; they are trapped in the terrestrial magnetic field.

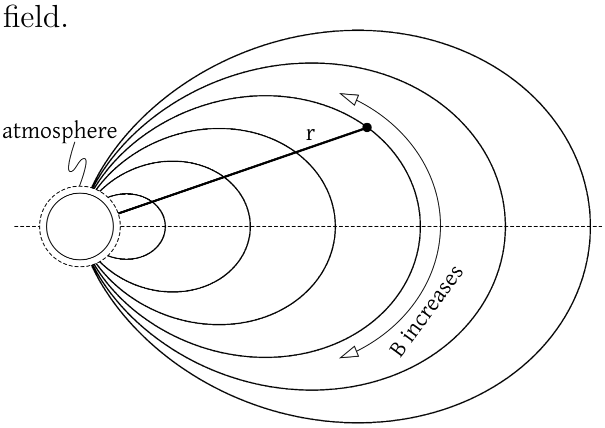

Figure 5: Terrestrial dipolar field lines.

0.7 Loss cone

At the mirror point, where $W_\parallel = 0$, the pitch angle becomes $\alpha_m = 90°$. Hence from (19) we see that

$$\frac{\sin^2\alpha}{1} = \frac{B}{B_m} \tag{20}$$

which is true anywhere along the trajectory (field line). For a terrestrial dipole field line, the maximum ratio of B/B_m is between the equatorial plane and the Earth's surface. For a field lines that crosses the equatorial plane at a distance of 4 Earth radii (R_e)

$$\alpha = \arcsin\sqrt{\frac{B_{eq}}{B_{sf}}} \simeq 5.5° \tag{21}$$

If α is smaller the particle will hit the Earth's surface before it has a chance to mirror, Therefore, all particles with $\alpha < 5.5°$ are lost; this defines the loss cone (for that particular field line at 4 R_e.

0.8 $\vec{E} \times \vec{B}$ drift

Now assume $\vec{E} \neq 0$ and $\vec{B} = B_0\vec{e}_z$. Then from (1)

$$\frac{dv_x}{dt} = \frac{q}{m}(E_x + v_y B_0) \tag{22}$$
$$\frac{dv_y}{dt} = \frac{q}{m}(E_y - v_x B_0) \tag{23}$$
$$\frac{dv_z}{dt} = \frac{q}{m}E_z \tag{24}$$

(24) describes simple accelerated motion along $\vec{B}$ (change in total kinetic energy!). For simplicity, assume $\vec{E} = E_y\vec{e}_y$:

$$\frac{dv_x}{dt} = \frac{q}{m}v_y B_0 \tag{25}$$
$$\frac{dv_y}{dt} = \frac{q}{m}(E_y - v_x B_0) \tag{26}$$

Differentiate and substitute:

$$\frac{d^2v_x}{dt^2} = -\left(\frac{qB_0}{m}\right)^2\left(v_x - \frac{E_y}{B_0}\right) \tag{27}$$
$$\frac{d^2v_y}{dt^2} = -\left(\frac{qB_0}{m}\right)^2 v_y, \tag{28}$$

where the additional E_y/B_0 term indicates a drift velocity in the $+x$ direction. A more general consideration gives

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}. \tag{29}$$

$\vec{v}_E$ is superposed on the gyromotion and independent of charge; i.e., electrons and protons drift in the same direction $\rightsquigarrow$ no net current.

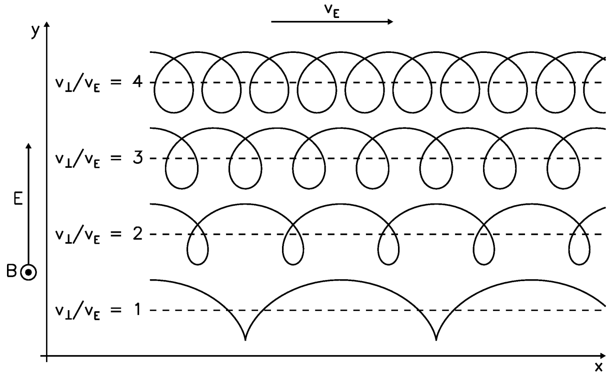

Figure 6: $\vec{E} \times \vec{B}$ drift motion for a positively charged particle.

0.9 Magnetic drifts

Two magnetic drifts exists, the gradient drift and the curvature drift. For the first, assume $\vec{E} = 0$ and $\vec{B}$ is spatially inhomogeneous, i.e., $\nabla B \neq 0$. In the region of large $\vec{B}$ the gyroradius is smaller than in the region of smaller $\vec{B}$. From this results a drift:

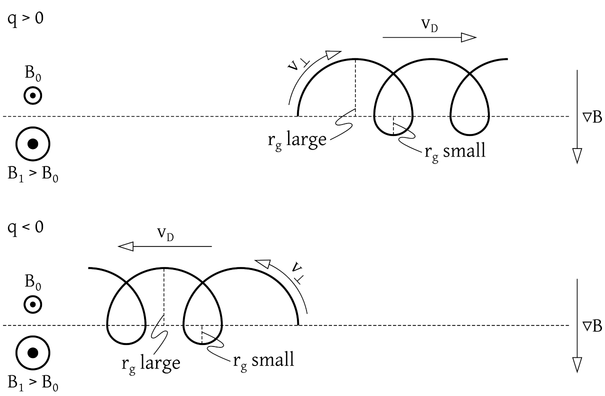

Figure 7: Illustration of the gradient drift for positively and negatively charged particles.

Formally

$$\vec{v}_G = \frac{mv_\perp^2}{2} \frac{\vec{B} \times \nabla B}{qB_0^3}, \qquad (30)$$

the drift velocity is charge dependent, i.e., the gradient drift causes a current. The gradient drift velocity depends on the perpendicular kinetic energy.

The second magnetic drift is the curvature drift. It is due to centrifugal forces experienced by a particle as it gyrates along a curved field line:

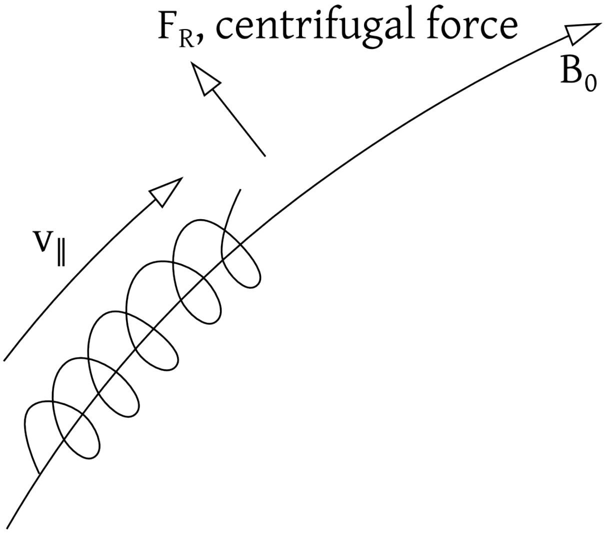

Figure 7: Illustration of the curvature drift.

The drift velocity is given by

$$\vec{v}_R = \frac{mv_\parallel^2}{2} \frac{2\vec{R}_c \times \vec{B}}{qR_c^2 B_0^2}, \qquad (31)$$

Again, the curvature drift is charge dependent, i.e., positively and negatively charged particles drift in opposite directions, creating a current. The velocity of the curvature drift depends on the parallel kinetic energy

0.10 General force drifts

Replacing $\vec{E}$ in (29) with $\vec{E} = \vec{F}/q$ one obtains:

$$\vec{v} = \frac{1}{\omega_g} \left(\frac{\vec{F}}{m} \times \frac{\vec{B}}{B} \right) \qquad (32)$$