Some aspects of kinetic theory

In the previous section we have learned how one individual particle moves under the influence of external electric and magnetic fields. However, plasmas are collections of MANY particles, all governed by their EoM:

$$m_i \frac{d\vec{v}_i}{dt} = \vec{F}_m = q_i(\vec{E}_m + \vec{v}_i \times \vec{B}_m) \ \ (+ \text{ gravity } + ...), \tag{1}$$

meaning that the motion of the ith particle is determined by the combined force $\vec{F}_m$ created by all other particles. The electromagnetic fields can be calculated using Maxwell's equations:

$$\nabla \cdot \vec{E}_m = \epsilon_0 \rho_m \tag{2}$$

$$\nabla \times \vec{E}_m = -\frac{\partial \vec{B}_m}{\partial t} \tag{3}$$

$$\nabla \cdot \vec{B}_m = 0 \tag{4}$$

$$\nabla \times \vec{B}_m = \mu_0 \vec{j}_m + \mu_0 \epsilon_0 \frac{\partial \vec{E}_m}{\partial t} \tag{5}$$

where the individual charged and moving particles form the sources for $\vec{j}_m$ and ρ_m:

$$\rho_m = \sum_s q_s \int f_s(\vec{r}, \vec{v}, t) d^3v \tag{6}$$

$$\vec{j}_m = \sum_s q_s \int f_s(\vec{r}, \vec{v}, t) \vec{v} d^3v \tag{7}$$

where the summation is over each species s of plasma particle.
Theoretically, this set of coupled partial differential equations is solvable if the initial positions and velocities are known for each particle (Laplace's demon). In practice, however, solving this set of equations is very difficult! Here we will try and derive a simpler set of equations.

0.1 Particle distribution functions

In (6) and (7) $f_s(\vec{r}, \vec{v}, t)$ denotes the particle distrubution function (PDF) which gives the amount of particles that are located at position $\vec{r}+d^3r$ and move with a velocity of $\vec{v}+d^3v$. In other words, f gives the particles position in the six dimensional phase space.

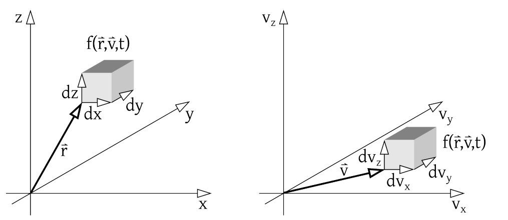

Figure 1: Sketch of the phase space and a distribtion function element.

0.1.1 Phase space

At every time we can find a particles position and velocity by locating it in the phase space. As the particle moves under the influence of external forces, it forms a trajectory.
Example: the pendulum

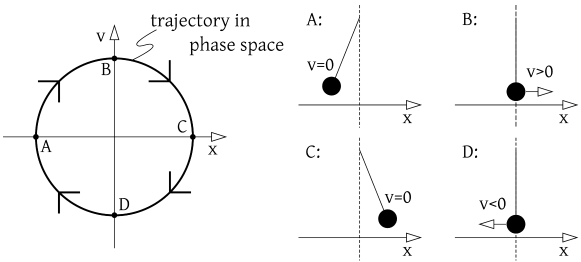

Figure 2: Trajectory of a pendulum in phase space.

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But instead of talking about the velocity of every particle, maybe is it more practical to talk about the average velocity? And instead of talking about particles at a certain location with a certain velocity, it is more practical to talk just about the number of particles, irrespective of the velocities?

0.2 Macroscopic variables

$f_s(\vec{r}, \vec{v}, t)$ gives the number of particles at a certain position with a certain velocity. In order to find the amount of particles at a certain position - the number density $n_s(\vec{r}, t)$ - sum (integrate) the PDF over all velocities:

$$\int f_s(\vec{r}, \vec{v}, t) d^3v = n_s(\vec{r}, t) \quad (8)$$

This integral is called the zeroth moment of the PDF; zero, because the integrand f_s is multiplied by $\vec{v}^0$. The first moment of the PDF, when the integrand is multiplied by $\vec{v}^1$ gives the average movement of these particles - the average number flux:

$$\int f_s(\vec{r}, \vec{v}, t) \vec{v} d^3v = n_s(\vec{r}, t) \vec{v}_{s,b}(\vec{r}, t) \quad (9)$$

or

$$\frac{\int f_s(\vec{r}, \vec{v}, t) \vec{v} d^3v}{\int f_s(\vec{r}, \vec{v}, t) d^3v} = \vec{v}_{s,b}(\vec{r}, t) \quad (10)$$

Integrating the PDF multiplied twice with the velocity (the second moment of the PDF) gives the pressure (tensor):

$$m_s \int f_s(\vec{r}, \vec{v}, t) \vec{v} \otimes \vec{v} d^3v = P_s(\vec{r}, t) \quad (11)$$

0.3 Conservation of phase space density

$f_s(\vec{r}, \vec{v}, t)$ describes the motion of particles within a physical system; if no particles are created or destroyed, then

$$\frac{d}{dt} f_s = 0 \quad (12)$$

or after expanding the total derivative and inserting $\vec{a}_s = q_s/m_s(\vec{E} + \vec{v} \times \vec{B})$ one obtains the Vlasow equation (VE)

$$\frac{\partial f_s}{\partial t} + \nabla_{\vec{r}} f_s \cdot \vec{v} + \nabla_{\vec{v}} f_s \cdot \frac{q_s}{m_s} (\vec{E} + \vec{v} \times \vec{B}) = 0 \quad (13)$$

0.4 Moments of the Vlasow equation

Again we integrate over all velocities to obtain the zeroth moment of the VE:

$$\int \left[\frac{\partial f_s}{\partial t} + \nabla_{\vec{r}} f_s \cdot \vec{v} + \nabla_{\vec{v}} f_s \cdot \frac{q_s}{m_s} (\vec{E} + \vec{v} \times \vec{B}) \right] d^3v = 0 \quad (14)$$

1. term:

$$\int \frac{\partial f_s}{\partial t} d^3v = \frac{\partial}{\partial t} \int f_s d^3v = \frac{\partial}{\partial t} n_s \quad (15)$$

2. term:

$$\int \nabla_{\vec{r}} f \cdot \vec{v} d^3v = \nabla_{\vec{r}} \cdot \int f \vec{v} d^3v = \nabla_{\vec{r}} \cdot (n_s \vec{v}_{s,b}) \quad (16)$$

3. term:

$$\int \nabla_{\vec{v}} f \cdot \frac{q_s}{m_s} (\vec{E} + \vec{v} \times \vec{B}) d^3v = 0 \quad (17)$$

As the first moment of the VE we obtain

$$\frac{\partial}{\partial t} n_s + \nabla_{\vec{r}} \cdot (n_s \vec{v}_{s,b}) = 0, \quad (18)$$

i.e., the continuity equation for the macroscopic variables.
Similarly we can compute the first moment of the VE which turns out to be

$$\frac{\partial (n_s \vec{v}_{s,b})}{\partial t} + \nabla_{\vec{r}} \cdot (n_s \vec{v}_{s,b} \vec{v}_{s,b}) + \frac{1}{m} \nabla_{\vec{r}} \cdot P_s - \frac{q_s}{m_s} n_s (\vec{E} + \vec{v}_{s,b} \times \vec{B}) = 0 \quad (19)$$

i.e., the equation of motion for the macroscopic variables.

0.5 One fluid theory

Usually, we have more than one species of particles in our plasma. At the very least, there are (negative) electrons and (positive) ions. That means that we need to deal with two EoM, two continuity equations, two number densities, etc. In this simplest case (two species) we can further simplify things by introducing averages of the macroscopic variables like:

$$n = \frac{m_e n_e + m_i n_i}{m_e + m_i} \tag{20}$$

$$m = m_e + m_i \tag{21}$$

$$\vec{v} = \frac{m_e n_e \vec{v}_{e,b} + m_i n_i \vec{v}_{i,b}}{m_e n_e + m_i n_i} \tag{22}$$

and adding the two continuity equations we get the one fluid version of the continuity equation (all ∇'s are now with respect to the spatial variable $\vec{r}$)

$$\frac{\partial}{\partial t} n + \nabla \cdot (n \vec{v}_b) = 0. \tag{23}$$

For the EoM we obtain

$$\frac{\partial (nm\vec{v}_b)}{\partial t} + \nabla \cdot (nm\vec{v}_b\vec{v}_b) = - \nabla \cdot P + \vec{j} \times \vec{B} + \rho \vec{E}. \tag{24}$$

The rest comes out as

$$\rho = e(n_i - n_e) \tag{25}$$

$$\vec{j} = e(n_i \vec{v}_{i,b} - n_e \vec{v}_{e,b}) \tag{26}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{27}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{28}$$

$$\nabla \cdot \vec{B} = 0 \tag{29}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{30}$$

(31)

Subtracting the ion and electron EoM we get the generalized Ohm's Law

$$\vec{E} = -\vec{v}_b \times \vec{B} + \vec{j}/\sigma - \frac{1}{en}\nabla p_e + \frac{1}{en}\vec{j} \times \vec{B} + \frac{m_e}{e^2}\frac{d}{dt}\left(\frac{\vec{j}}{n}\right) \tag{32}$$

0.6 Magnetohydrodynamics

Space plasmas are usually quasi-neutral, i.e., $n_i = n_e$. It follows that there are no space charges $\rho = 0$ and hence the only source for electric fields are temporal changes in the magnetic field. If you also consider slow (relative to the plasma frequency) and large (relative to the Debye length) phenomena, we find that

$$\frac{|\mu_0 \vec{j}|}{\left|\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right|} \gg 1 \tag{33}$$

such that we can neglect the displacement current in Ampere's Law. It also turns out that Ohm's Law can be simplified and you obtain

$$\frac{\partial}{\partial t} n + \nabla \cdot (n\vec{v}_b) = 0 \tag{34}$$

$$\frac{\partial (nm\vec{v}_b)}{\partial t} + \nabla \cdot (nm\vec{v}_b\vec{v}_b) = -\nabla \cdot P + \vec{j} \times \vec{B} \tag{35}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{36}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} \tag{37}$$

$$\vec{j} = \sigma(\vec{E} + \vec{v}_b \times \vec{B}) \tag{38}$$

0.7 Phenomenological derivation of the continuity equation

In the previous sections we have derived from first principles a set of equations that describes the dynamics of plasmas under several assumptions. The basis for the derivation was the collective particle behaviour as described by the PDF. There is, however, a phenomenological approach which is more ad hoc but at the same time it might be more intuitive. Consider particles in a volume V with a bulk velocity $\vec{v}$ and density n.

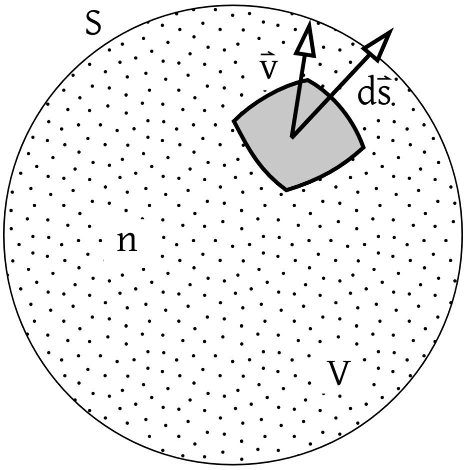

Figure 3: Sketch of the volume V with surface S for which the continuity equation is derived.

Then

$$N = \int_V n dV. \tag{39}$$

The flux of particles through a surface element $d\vec{s}$ into and out of this volume is

$$f_S = n\vec{v} \cdot d\vec{s} \tag{40}$$

and hence the flux through the entire surface is

$$F = \oint_S n\vec{v} \cdot d\vec{s}. \tag{41}$$

A change in N or V must be equal to a flux F (if no particles are destroyed or created).

$$\frac{\partial N}{\partial t} = -F \tag{42}$$

$$\frac{\partial}{\partial t} \int_V n dV = -\oint_S n\vec{v} \cdot d\vec{s} \tag{43}$$

$$= -\int_V \nabla \cdot (n\vec{v}) dV \tag{44}$$

and hence follows the continuity equation

$$\int_V \left[\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) \right] = 0. \tag{45}$$