1 Aspects of MHD

Remember the MHD equations, now introducing the mass density as ρ, assuming the pressure is a scalar p, and dropping the index b from the bulk velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \nabla)\vec{v} = -\nabla p + \vec{j} \times \vec{B} \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} \tag{4}$$

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \tag{5}$$

We are now going to explore some of the important consequences that can be derived from these equations.

1.1 Magnetic diffusion and frozen flux

Combine Ohm's Law with Faraday's Law and Ampére's Law assuming that the condutivity σ is constant:

$$\nabla \times \vec{E} = \nabla \times \left(\vec{j}/\sigma - \vec{v} \times \vec{B}\right) = \frac{1}{\mu_0 \sigma} \nabla \times \nabla \times \vec{B} - \nabla \times (\vec{v} \times \vec{B}) = -\frac{\partial \vec{B}}{\partial t}. \tag{6}$$

After some manipulation we arrive at the induction equation

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \vec{B}, \tag{7}$$

where the first term on teh right hand side is called the convection term and the second term is the diffusion term. The ratio of the two terms lets us estimate which process dominates and defines the magnetic Reynolds number R_m. In the estimation, we use $\nabla \sim 1/L$ where L is a typical scale length of the system under consideration

$$R_m \approx \frac{vB/L}{\frac{1}{\mu_0 \sigma} B/L^2} = \mu_0 \sigma v L \tag{8}$$

Two extreme cases are possible

- If $R_m \gg 1$, then convection dominates ($\sigma \to \infty$)
- If $R_m \ll 1$, then diffusion dominates ($\sigma \to 0$)

1.2 Frozen-in flux

Consider a parcel of plasma moving through a magnetic field:

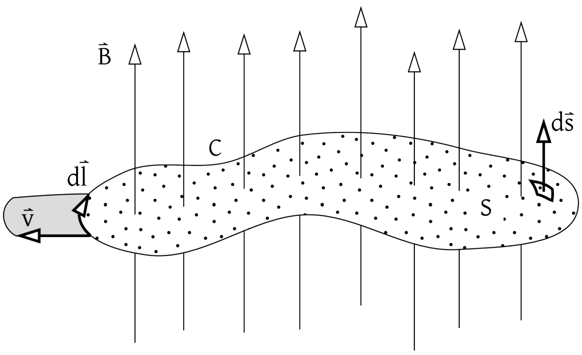

Figure 1: A patch of plasma moving through a magnetic field.

The magnetic flux through the surface S is given by

$$\Psi = \int_S \vec{B} \cdot d\vec{s} \tag{9}$$

Ψ can change due to (1) a change in the magnetic field through S and (2) due to a change of B along the path of S:

$$\left(\frac{\partial \Psi}{\partial t}\right)_1 = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \quad (10)$$

$$\left(\frac{\partial \Psi}{\partial t}\right)_2 = \oint_C \vec{B} \cdot \vec{v} \times d\vec{l} \quad (11)$$

$$= \oint_C \vec{B} \times \vec{v} \cdot d\vec{l} \quad (12)$$

$$= \int_S \nabla \times (\vec{B} \times \vec{v}) \cdot d\vec{s}. \quad (13)$$

Therefore

$$\frac{d\Psi}{dt} = \int_S \left[\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B})\right] \cdot d\vec{s} \quad (14)$$

which is the induction equation with the diffusion term neglected. The change in magnetic flux through the surface S is only in such conditions constant, where there is no magnetic diffusion, i.e., the magnetic field cannot change without moving the plasma and the plasma cannot move without changing the magnetic field. The plasma and the field are frozen-in. This condition occurs for high magnetic Reynolds numbers, in other words $d\Psi/dt = 0$ if $R_m \gg 1$ or $\sigma \to \infty$.
If, on the other hand, $R_m \ll 1$ or $\sigma \to 0$ then the motion of the plasma and the change in magnetic field are decoupled, allowing the magnetic field to diffuse out of a region of plasma making $d\Psi/dt \neq 0$.

1.3 Magnetic pressure and tension

Explore the $\vec{j} \times \vec{B}$ term in the MHD EoM:

$$\vec{j} \times \vec{B} = \left(\frac{1}{\mu_0}\nabla \times \vec{B}\right) \times \vec{B} \quad (15)$$

$$= \frac{1}{\mu_0}(\vec{B} \cdot \nabla)\vec{B} - \nabla\left(\frac{B^2}{2\mu_0}\right). \quad (16)$$

Substitute into EoM

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \nabla)\vec{v} = -\nabla p - \nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\vec{B} \cdot \nabla)\vec{B}. \quad (17)$$

The last term on the right hand side describes forces acting on the plasma due to changes of $\vec{B}$ along the direction of $\vec{B}$. In simple terms, magnetic fields lines want to be as straight as possible and when they are kinked, there acts a force trying to straighten the field line. After realizing that $B^2/2\mu_0$ has units of pressure, we can combine the first the gradient of the thermal pressure p and the magnetic pressure p_B to that of the total pressure p_T.

1.4 Plasma beta

The ratio of thermal to magnetic pressure is called the plasma beta:

$$\beta = \frac{2\mu_0 p}{B^2} \quad (18)$$

The plasma beta characterizes the plasma; if $\beta \ll 1$ then the magnetic pressure dominates and the magnetic field determines the motion of the plasma. If $\beta \gg 1$ then the thermal pressure dominates and the plasma takes the magnetic field with it as it moves.

1.5 Magnetic reconnection

The frozen-in theorem states that plasma elements connected to any one field line at any point in time always stay connected to that very field line. However, there is evidence that this principle is sometimes broken. The agent that facilitates the reorganization of plasma and magnetic field is called magnetic reconnection.

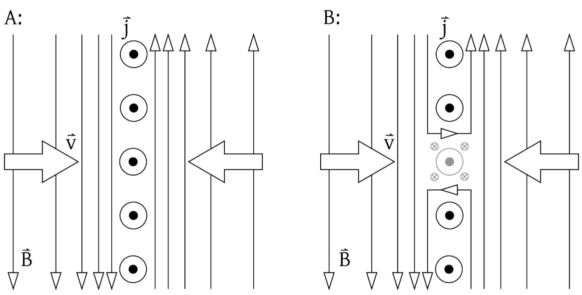

Figure 2: Schematic overview of magnetic reconnection.

Consider a situation where magnetic field is (because it is frozen into the plasma) convected toward a boundary from two sides with a plasma flow, as indicated in Figure 2A. If the magnetic fields on the right and on the left are anti-parallel, there has to exist a current sheet between the magnetic field shear (Ampére Law). As more and more magnetic field is brought to the boundary, the currents increase up until a point where the plasma cannot support the current necessary to uphold the magnetic shear. Locally, due to a plasma instability, the current breaks down, or, equivilantly, a counter current is created. The is indicated in gray in Figure 2B. The superposition of the original (sheet) current and the locally created counter current gives rise to a new magnetic field topology where now field lines from the left region are now connected to the right region. Because locally the frozen-in theorem has to be violated and the current breaks down, one sometimes speaks of anomalous resistivity at the reconnection region.

1.6 Sweet-Parker model

The Sweet-Parker model tries to explain the phycics of reconnection; it is a model that is steady-state ($\partial/\partial t = 0$), two dimensional ($\partial/\partial z = 0$). It also assumes incompressible flows, i.e., $\rho = \rho_i = \rho_o$.

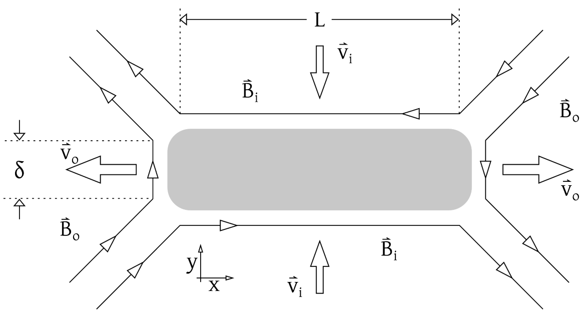

Figure 3: Schematic of the Sweet-Parker model.

Consider the situation in Figure 3 where the magnetic field is brought to the reconnection region (gray shaded area) over a large length L by a plasma flow v_i. The mass flow into the reconnection region is hence $\rho v_i L$ and this mass flow has to be conserved, i.e., the mass outflow from the reconnection region across the much smaller scale δ is $\rho v_o \delta$ and $\rho v_i L = \rho v_o \delta$, or

$$\frac{v_i}{v_o} = \frac{\delta}{L} \ll 1. \tag{19}$$

Energy is of course also conserved such that the kinetic and electromagnetic energy flow into the reconnection region has to be balanced by the kinetic and electromagnetic energy flow out of the reconnection region, such that (remember from ideal MHD Ohm's Law $\vec{E} = -\vec{v} \times \vec{B}$)

$$\left(\frac{1}{2}\rho v_i^2 v_i + \frac{1}{\mu_0} v_i B_i^2\right) L = \left(\frac{1}{2}\rho v_o^2 v_o + \frac{1}{\mu_0} v_o B_o^2\right) \delta. \tag{20}$$

Using the results from mass conservation we obtain

$$\frac{\rho v_i^2}{2} + \frac{B_i^2}{\mu_0} = \frac{\rho v_o^2}{2} + \frac{B_o^2}{\mu_0}. \tag{21}$$

We have already seen that $v_i \ll v_o$; it is also intuitively clear that in the inflow region the magnetic field is piling up and therefore $B_i \gg B_o$ and hence

$$\frac{B_i^2}{\mu_0} \approx \frac{\rho v_o^2}{2}, \tag{22}$$

or in terms of magnetic and kinetic pressure

$$2 p_{B,i} \approx p_{k,o}. \tag{23}$$

This then means that the magnetic pressure, or magnetic energy density, from the inflow region is converted through reconnection into kinetic energy density (kinetic pressure) of the plasma in the outflow region. Magnetic reconnection converts energy stored in magnetic fields into kinetic energy of the plasma.
Whereas the Sweet-Parker model gives some pleasing results about the scaling relations between magnetic and kinetic energy, it fails to predict the high rates of reconnection observed in real life.