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Surface wave-induced drift

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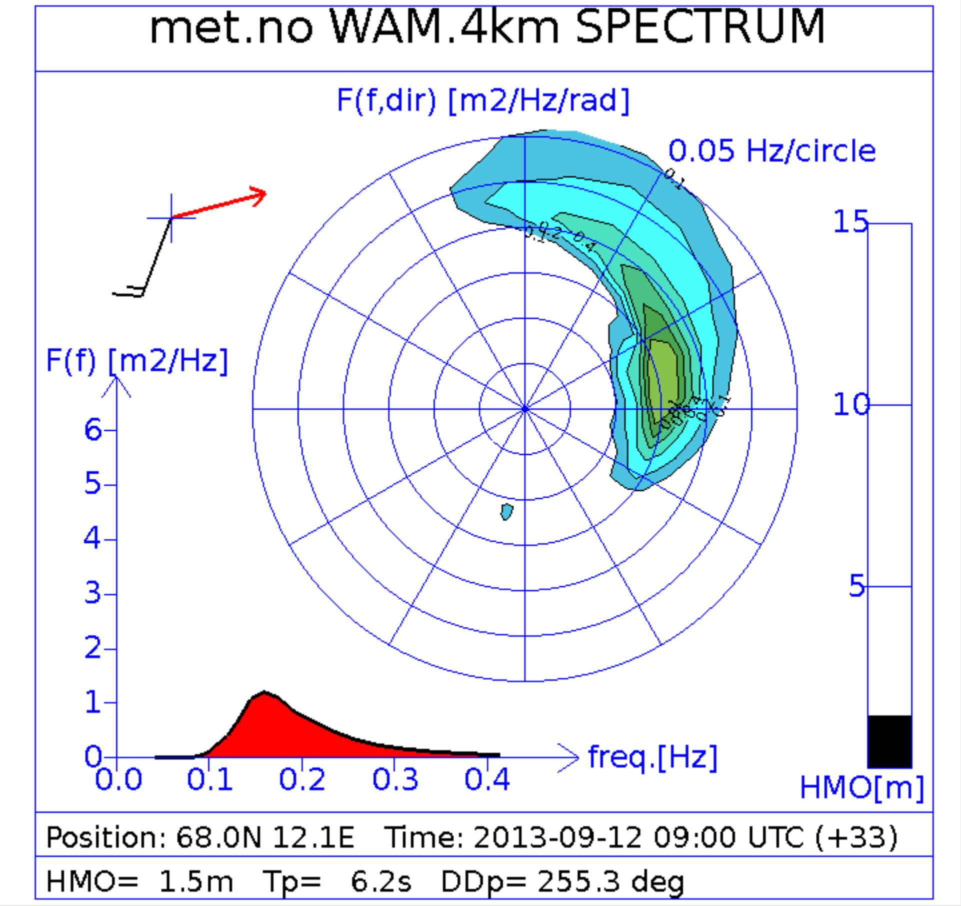
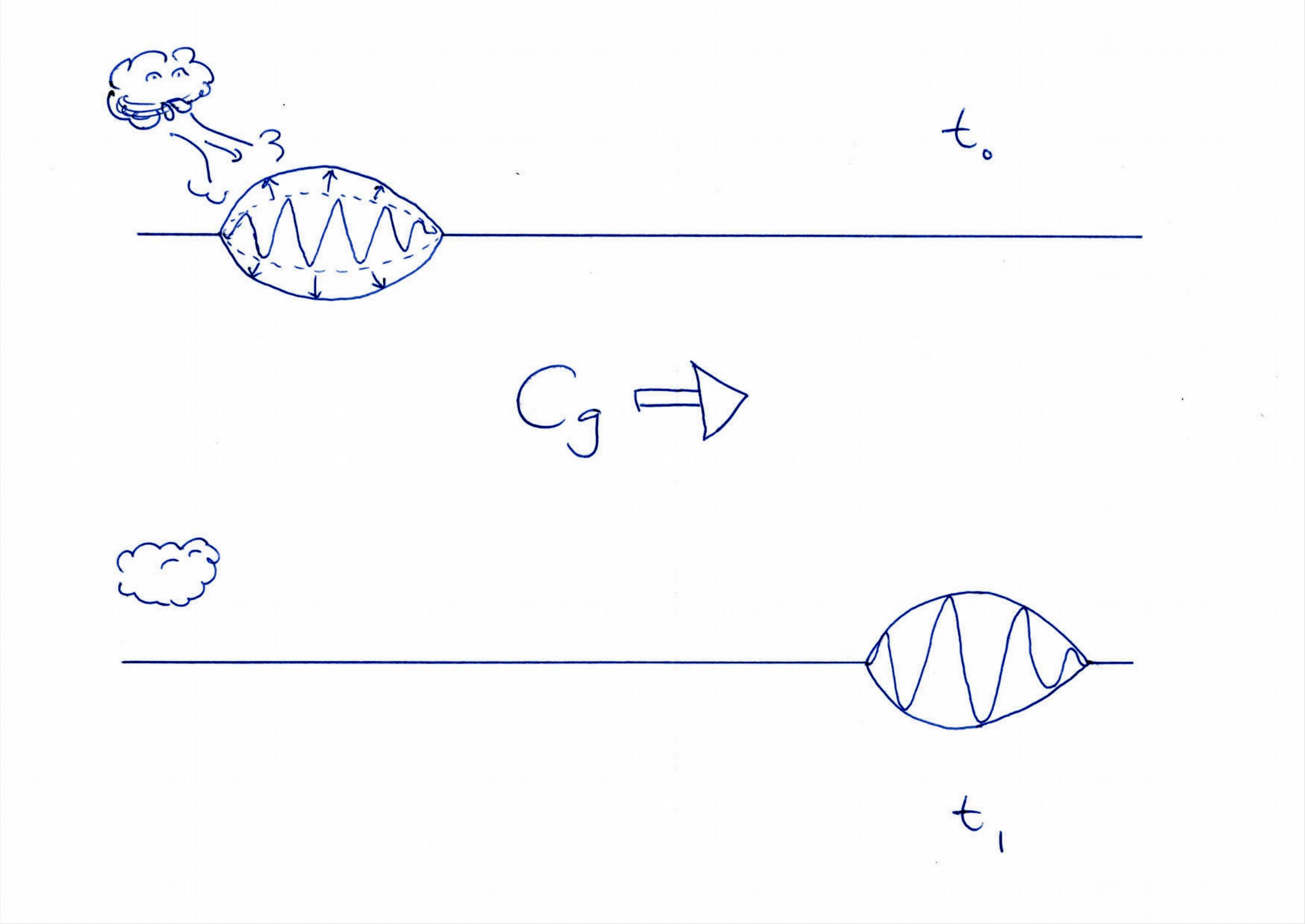
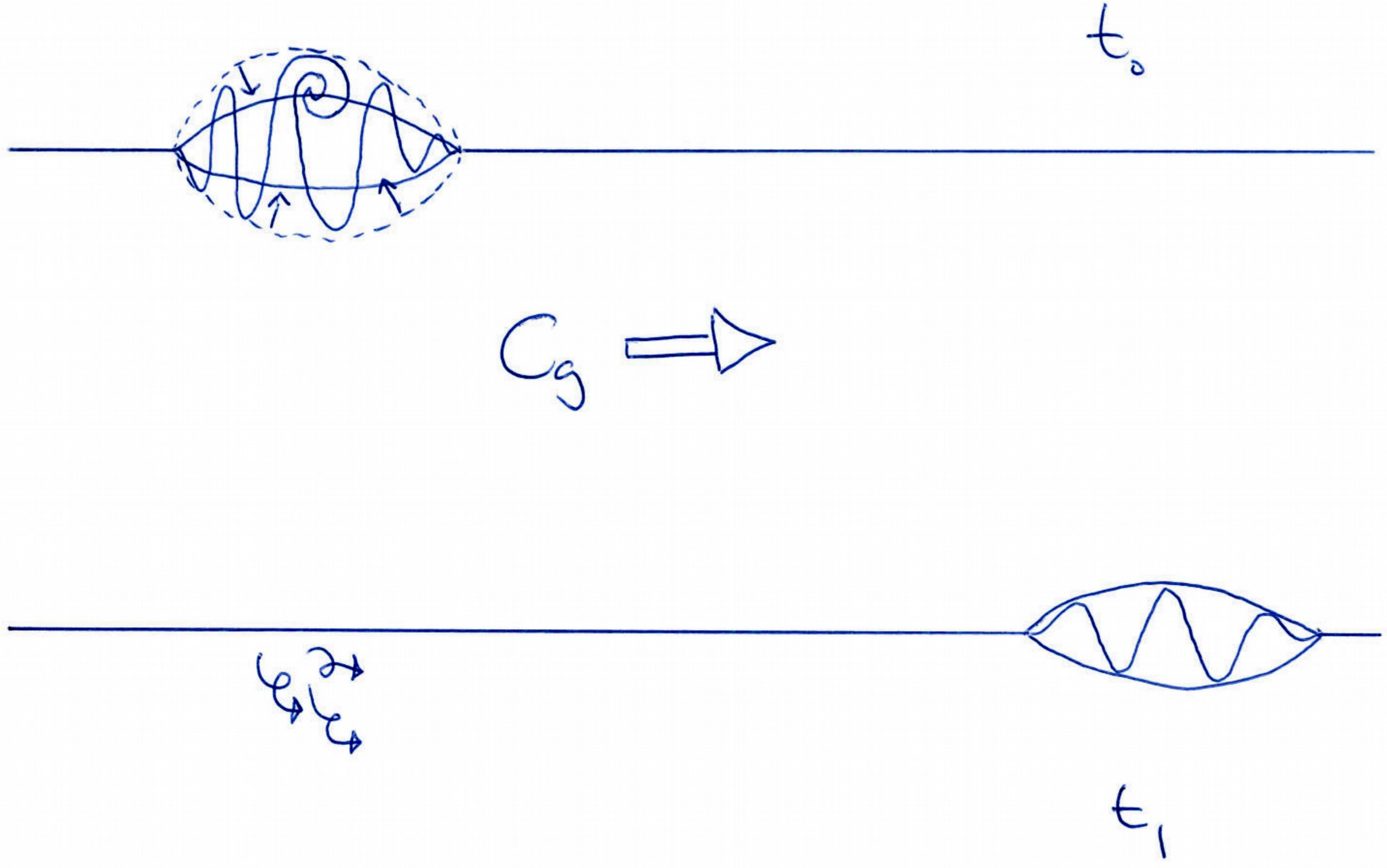
The energy balance equation

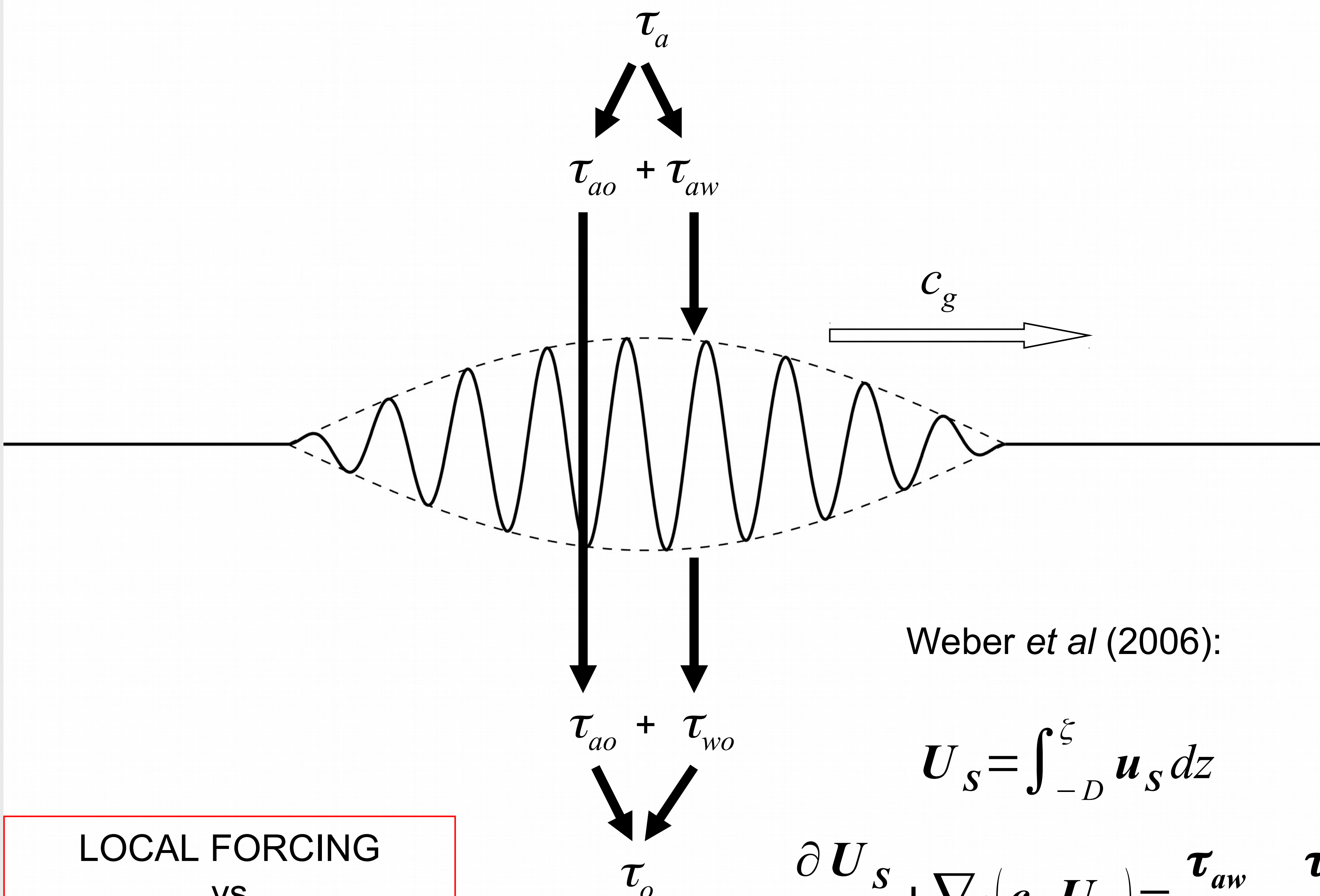
$$\frac{\partial F}{\partial t} + \nabla \cdot \left(\boldsymbol{c}_{\boldsymbol{g}} F \right) = S_{input} + S_{dissip} + S_{nonlin}$$





t_0
C_g
t_1


t0
Cg
t1



Weber *et al* (2006):

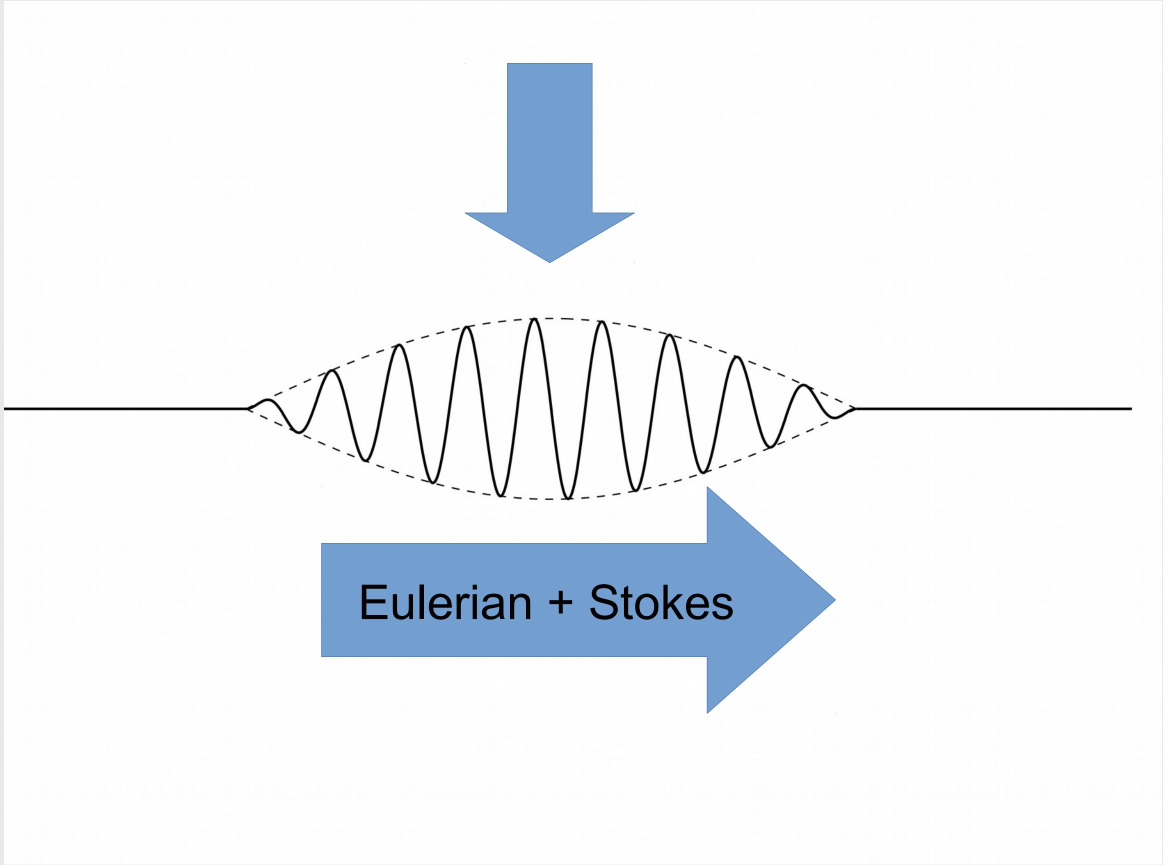
$$U_s = \int_{-D}^{\zeta} u_s \, dz$$

$$\frac{\partial U_s}{\partial t} + \nabla \cdot \left(c_g U_s \right) = \frac{\tau_{aw}}{\rho} - \frac{\tau_{wo}}{\rho}$$

LOCAL FORCING
vs
ADVECTION BY GROUP

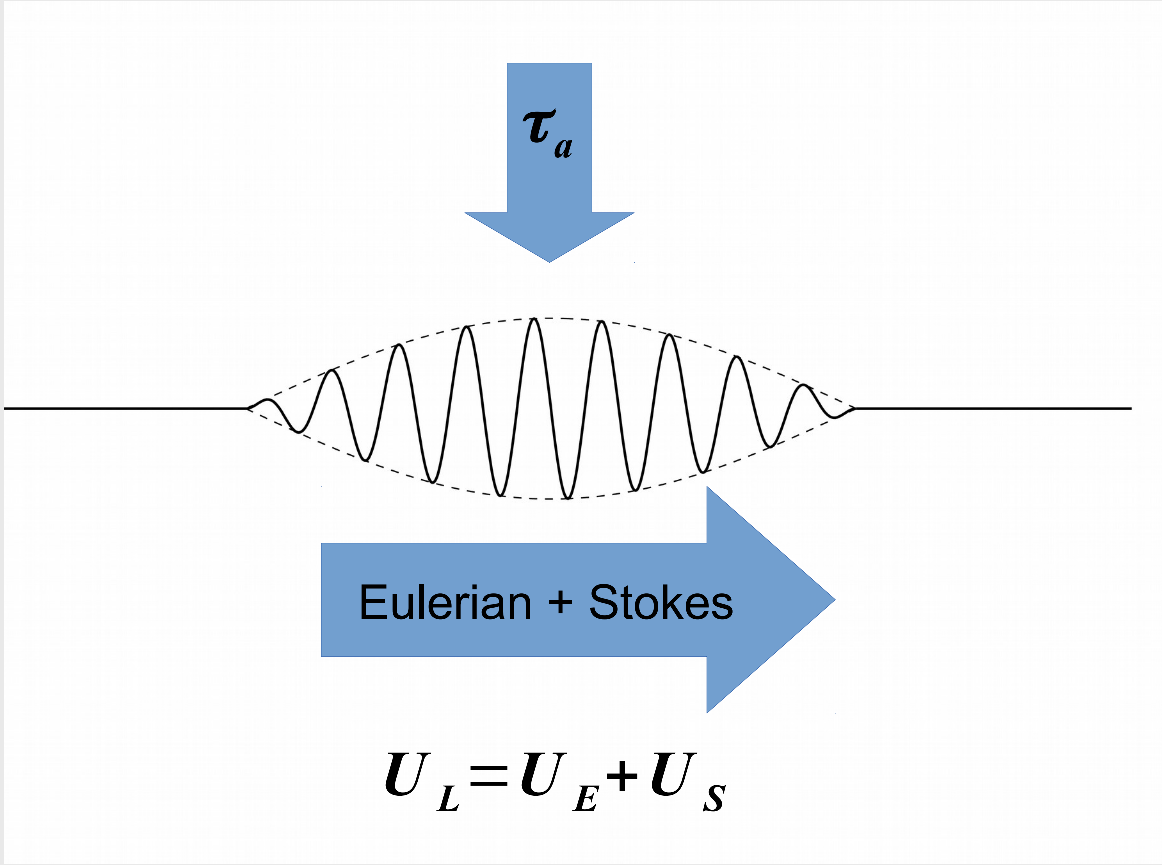
Ekman dynamics with waves

Momentum flux from atmosphere



Ekman dynamics with waves

$$\frac{\partial \boldsymbol{U}_L}{\partial t} + f\,\boldsymbol{k} \times \boldsymbol{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right)$$



Ekman dynamics with waves

$$\frac{\partial \boldsymbol{U}_L}{\partial t} + f\,\boldsymbol{k}\times\boldsymbol{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla\cdot\left(\boldsymbol{c}_g \boldsymbol{U}_S\right)$$

TOTAL MOMENTUM* IN UPPER OCEAN

$$\boxed{\boldsymbol{U}_L} = \boldsymbol{U}_E + \boldsymbol{U}_S$$

* Per unit density

Ekman dynamics with waves

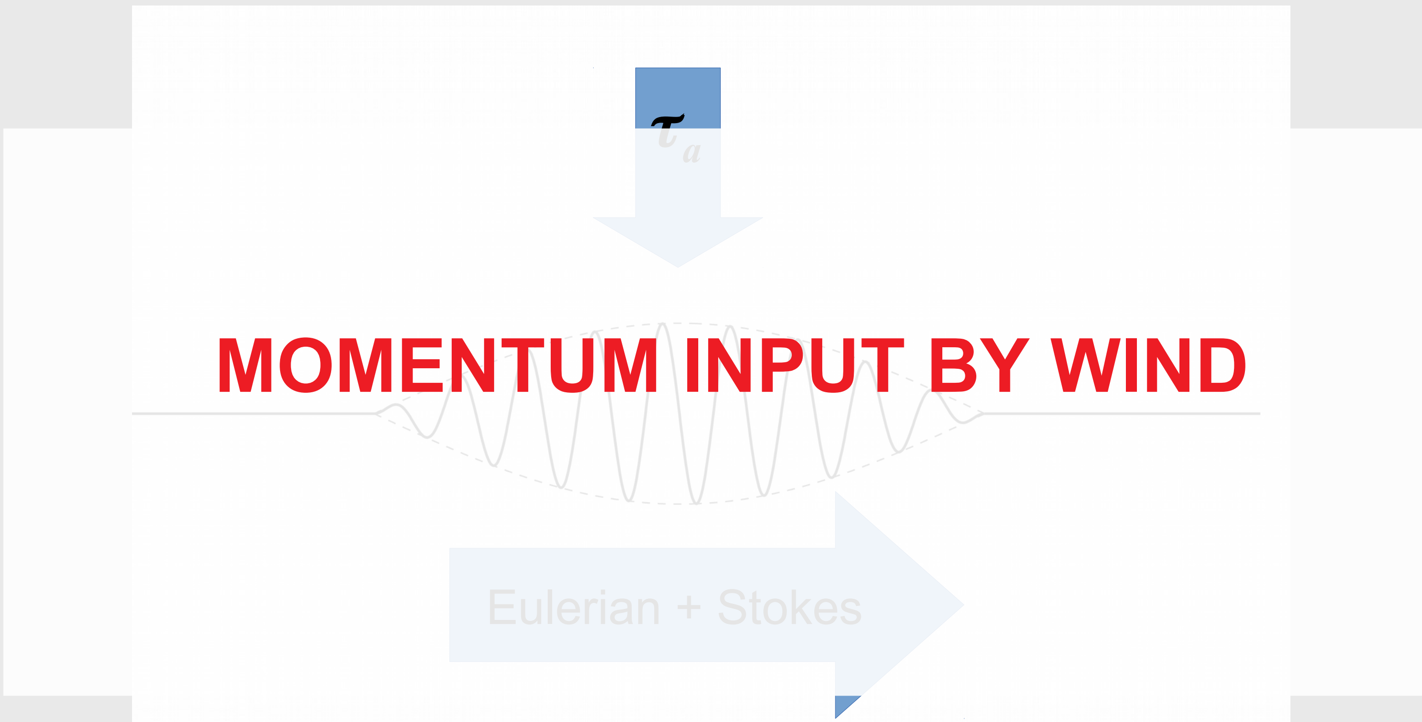
$$\frac{\partial \boldsymbol{U}_L}{\partial t} + f\,\boldsymbol{k}\times\boldsymbol{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla\cdot\left(\boldsymbol{c}_g\boldsymbol{U}_S\right)$$

ROTATING REFENCE FRAME

$$\boldsymbol{U}_L = \boldsymbol{U}_E + \boldsymbol{U}_S$$

Ekman dynamics with waves

$$\frac{\partial \boldsymbol{U}_L}{\partial t} + f\,\boldsymbol{k} \times \boldsymbol{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right)$$

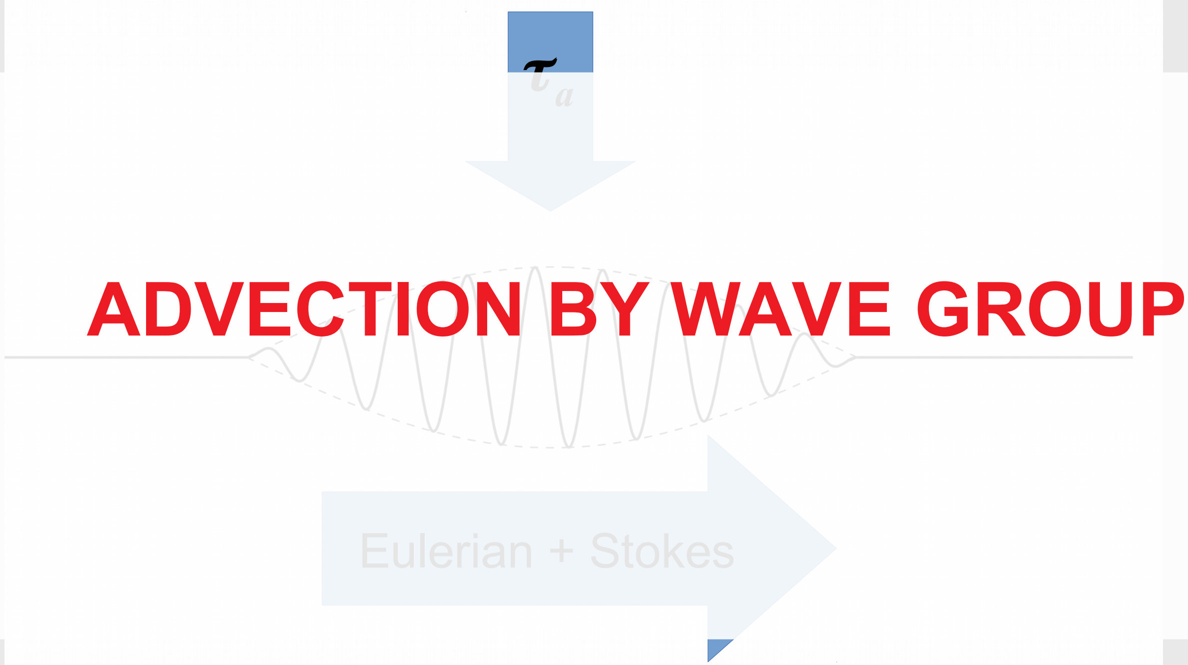


MOMENTUM INPUT BY WIND

$$\boldsymbol{U}_L = \boldsymbol{U}_E + \boldsymbol{U}_S$$

Ekman dynamics with waves

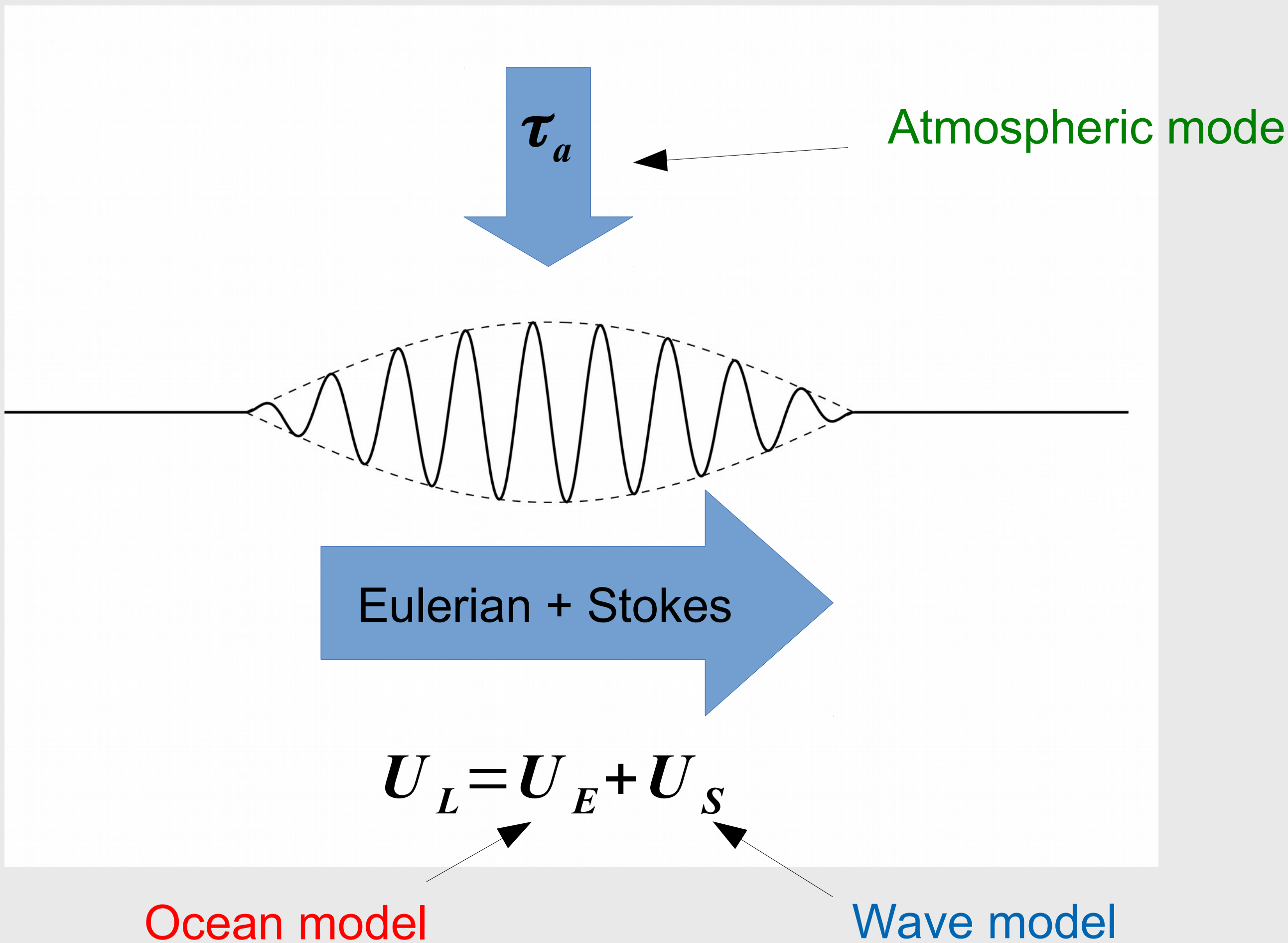
$$\frac{\partial \boldsymbol{U}_L}{\partial t} + f\,\boldsymbol{k} \times \boldsymbol{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right)$$



$$\boldsymbol{U}_L = \boldsymbol{U}_E + \boldsymbol{U}_S$$

Ekman dynamics with waves

$$\frac{\partial \boldsymbol{U}_L}{\partial t} + f\,\boldsymbol{k}\times\boldsymbol{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla\cdot\left(\boldsymbol{c}_g \boldsymbol{U}_S\right)$$



$$\frac{\partial \boldsymbol{U}_L}{\partial t} + f\,\boldsymbol{k} \times \boldsymbol{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right)$$

(expanding)

$$\frac{\partial \boldsymbol{U}_E}{\partial t} + \frac{\partial \boldsymbol{U}_S}{\partial t} + f\,\boldsymbol{k} \times \left(\boldsymbol{U}_E + \boldsymbol{U}_S \right) = \frac{\boldsymbol{\tau}_{ao}}{\rho} + \frac{\boldsymbol{\tau}_{aw}}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right)$$

$$\frac{\partial \boldsymbol{U}_L}{\partial t} + f\,\boldsymbol{k} \times \boldsymbol{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right)$$

(expanding)

$$\frac{\partial \boldsymbol{U}_E}{\partial t} + \frac{\partial \boldsymbol{U}_S}{\partial t} + f\,\boldsymbol{k} \times \left(\boldsymbol{U}_E + \boldsymbol{U}_S \right) = \frac{\boldsymbol{\tau}_{ao}}{\rho} + \frac{\boldsymbol{\tau}_{aw}}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right)$$

(rewriting the «wave model»)

$$\frac{\partial \boldsymbol{U}_S}{\partial t} + \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right) = \frac{\boldsymbol{\tau}_{aw}}{\rho} - \frac{\boldsymbol{\tau}_{wo}}{\rho} \Rightarrow \frac{\boldsymbol{\tau}_{aw}}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right) = \frac{\boldsymbol{\tau}_{wo}}{\rho} + \frac{\partial \boldsymbol{U}_S}{\partial t}$$

$$\frac{\partial \mathbf{U}_L}{\partial t} + f\,\mathbf{k} \times \mathbf{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla \cdot \left(\mathbf{c}_g \mathbf{U}_S \right)$$

(expanding)

$$\frac{\partial \mathbf{U}_E}{\partial t} + \frac{\partial \mathbf{U}_S}{\partial t} + f\,\mathbf{k} \times \left(\mathbf{U}_E + \mathbf{U}_S \right) = \frac{\boldsymbol{\tau}_{ao}}{\rho} + \frac{\boldsymbol{\tau}_{aw}}{\rho} - \nabla \cdot \left(\mathbf{c}_g \mathbf{U}_S \right)$$

(rewriting the «wave model»)

$$\frac{\partial \mathbf{U}_S}{\partial t} + \nabla \cdot \left(\mathbf{c}_g \mathbf{U}_S \right) = \frac{\boldsymbol{\tau}_{aw}}{\rho} - \frac{\boldsymbol{\tau}_{wo}}{\rho} \quad \Longrightarrow \quad \frac{\boldsymbol{\tau}_{aw}}{\rho} - \nabla \cdot \left(\mathbf{c}_g \mathbf{U}_S \right) = \frac{\boldsymbol{\tau}_{wo}}{\rho} + \frac{\partial \mathbf{U}_S}{\partial t}$$

(substitute)

$$\frac{\partial \mathbf{U}_E}{\partial t} + \frac{\partial \mathbf{U}_S}{\partial t} + f\,\mathbf{k} \times \left(\mathbf{U}_E + \mathbf{U}_S \right) = \frac{\boldsymbol{\tau}_{ao}}{\rho} + \frac{\boldsymbol{\tau}_{wo}}{\rho} + \frac{\partial \mathbf{U}_S}{\partial t}$$

$$\frac{\partial \boldsymbol{U}_L}{\partial t} + f\,\boldsymbol{k} \times \boldsymbol{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right)$$

(expanding)

$$\frac{\partial \boldsymbol{U}_E}{\partial t} + \frac{\partial \boldsymbol{U}_S}{\partial t} + f\,\boldsymbol{k} \times \left(\boldsymbol{U}_E + \boldsymbol{U}_S \right) = \frac{\boldsymbol{\tau}_{ao}}{\rho} + \frac{\boldsymbol{\tau}_{aw}}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right)$$

(rewriting the «wave model»)

$$\frac{\partial \boldsymbol{U}_S}{\partial t} + \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right) = \frac{\boldsymbol{\tau}_{aw}}{\rho} - \frac{\boldsymbol{\tau}_{wo}}{\rho} \quad \Rightarrow \quad \frac{\boldsymbol{\tau}_{aw}}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right) = \frac{\boldsymbol{\tau}_{wo}}{\rho} + \frac{\partial \boldsymbol{U}_S}{\partial t}$$

(substitute)

$$\frac{\partial \boldsymbol{U}_E}{\partial t} + \frac{\partial \boldsymbol{U}_S}{\partial t} + f\,\boldsymbol{k} \times \left(\boldsymbol{U}_E + \boldsymbol{U}_S \right) = \frac{\boldsymbol{\tau}_{ao}}{\rho} + \frac{\boldsymbol{\tau}_{wo}}{\rho} + \frac{\partial \boldsymbol{U}_S}{\partial t}$$

(simplify and use definition for ocean stress)

$$\frac{\partial \boldsymbol{U}_E}{\partial t} + f\,\boldsymbol{k} \times \left(\boldsymbol{U}_E + \boldsymbol{U}_S \right) = \frac{\boldsymbol{\tau}_o}{\rho}$$

$$\frac{\partial \boldsymbol{U}_L}{\partial t} + f\,\boldsymbol{k} \times \boldsymbol{U}_L = \frac{\boldsymbol{\tau}_a}{\rho} - \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right)$$

Summary: We take the momentum budget for the **mean flow and the waves combined**, and then *remove the parts specific to the «wave model»*. We are then left with the **«ocean model»**.

Coriolis-Stokes force

$$\frac{\partial \boldsymbol{U}_E}{\partial t} + f\,\boldsymbol{k} \times \left(\boldsymbol{U}_E + \boldsymbol{U}_S \right) = \frac{\boldsymbol{\tau}_o}{\rho}$$

Sea-state dependent momentum flux (surface stress)

Tying it all together

$$\frac{\boldsymbol{\tau}_o}{\rho} = \frac{\boldsymbol{\tau}_a}{\rho} - \left[\frac{\partial \boldsymbol{U}_S}{\partial t} + \nabla \cdot \left(\boldsymbol{c}_g \boldsymbol{U}_S \right) \right]$$

$$\frac{\partial F}{\partial t} + \nabla \cdot \left(\boldsymbol{c}_g F \right) = S_{input} + S_{dissip} + S_{nonlin}$$

$$\boldsymbol{U}_S = g \frac{F}{\boldsymbol{c}}$$

$$\boldsymbol{\tau}_o = \boldsymbol{\tau}_a - \rho g \int_0^{\infty} \int_0^{2\pi} \frac{\boldsymbol{k}}{\omega} \left(S_{input} + S_{dissip} \right) d\theta df$$

Notes

- Conceptual model!
  - Horizontal gradients are ignored
  - Finite depth effects ignored
  - Variable density is ignored
  - Vertical variation not described
  - Coupling ocean -> wave ignored
  - ...
- Spectral resolution of wave model can be an issue
- Similar coupling between the waves and the atmosphere, important process for weather forecasting