Chapter 6: The equations of fluid motion

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Ch. 6 - The equations of fluid motions

1. Motivation

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   3.1 Forces
   3.2 Equations of motion
   3.3 Hydrostatic balance

4. Conversation of mass

5. Thermodynamic equation

6. Equation of motion for a rotating fluid
   6.1 Forces
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7. Take home messages

*Addition, not in book.
1. Motivation

Motivation

- Derivation of the general equation of motion
- Different horizontal wind forms
- General circulation of atmosphere/ocean

Equations of fluid motions

• **Hydrodynamics** focuses on moving liquids and gases.

• **Hydrodynamics** and **thermodynamics** are the **foundations** of meteorology and oceanography.

• **Equations of motion** for the **atmosphere and ocean** are constituted by macroscopic conservation laws for substances, impulses and internal energy (hydrodynamics and thermodynamics).

  The characteristics of very small volume elements are looked at (e.g. overall mass, average speed).

  **Derivation of hydrodynamics based on the conservation of momentum, mass and energy.**

*hydro (greek): water*
# State variables for atmosphere and ocean

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>Wind ((u, v, w))</td>
<td>m/s</td>
</tr>
<tr>
<td>( T )</td>
<td>Temperature</td>
<td>°C or K</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Pressure</td>
<td>N/m(^2) = 1 Pa (1 hPa = 10(^2) Pa = 1 mbar)</td>
</tr>
</tbody>
</table>

Specifically for the **atmosphere**:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q )</td>
<td>Specific humidity</td>
<td>kg water vapour/kg moist air</td>
</tr>
</tbody>
</table>

Specifically for the **ocean**:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>Salinity</td>
<td>kg salt/kg seawater</td>
</tr>
</tbody>
</table>

(often the salt content is expressed in practical salinity unit: psu = kg salt/1,000 kg seawater).
Differentiation following the motion -
Euler and Lagrange perspectives

**Euler perspective**: The control volume is anchored stationary within the coordinate system, e.g. a cube through which the air or the seawater flows through.

**Lagrange perspective**: The control volume is an air or seawater parcel that always consists of the same particles and which moves along with the wind or the current.

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2. Basics
Differentiation following the motion - Euler – Lagrange derivatives

Example: wind blows over hill, cloud forms at the ridge of the lee wave; steady state assumption eq. cloud does not change in time.

**Eulerian derivative** (after Euler; 1707-1783):

\[
\left( \frac{\partial C}{\partial t} \right)_{\text{fixed point in space}} = 0,
\]

**Lagrangian derivative** (after Lagrange; 1736-1813):

\[
\left( \frac{\partial C}{\partial t} \right)_{\text{fixed particle}} \neq 0, \quad \frac{\partial}{\partial t}: \text{partial derivatives; other variables are kept fixed during the differentiation}
\]

\[ C = C(x,y,z,t): \text{cloud amount} \]
Euler and Lagrangian differentiation

Think about examples in your everyday life for Euler’s and Lagrange’s differentiation. Give three examples.
Differentiation following the motion mathematically

For arbitrary small variations:

\[
\delta C = \frac{\partial C}{\partial t} \delta t + \frac{\partial C}{\partial x} \delta x + \frac{\partial C}{\partial y} \delta y + \frac{\partial C}{\partial z} \delta z
\]

\((\delta C)_{\text{fixed particle}} = \left( \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} \right) \delta t\)

Dividing by \(\delta t\) and in limit of small variations:

\[
\left( \frac{\partial C}{\partial t} \right)_{\text{fixed particle}} = \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = \frac{DC}{Dt}
\]

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla
\]
2. Basics

Differentiation following the motion mathematically

Lagrangian or total derivative (after Lagrange; 1736-1813):

\[
\left( \frac{D}{Dt} \right)_{\text{fixed particle}} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}
\]

\[
\equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad (6-1)
\]

"Time rate of change of some characteristic of a particular element of fluid, which in general is changing its position."

Eulerian derivative (after Euler; 1707-1783):

\[
\left( \frac{\partial}{\partial t} \right)_{\text{fixed point}}
\]

"Time rate of change of some characteristic at a fixed point in space but with constantly changing fluid element because the fluid is moving."
Nomenclature summary

- $\delta$: Greek small delta, infinitesimal small size, e.g. $\delta V:= \text{small air volume}$. 
- $\frac{\partial}{\partial x}$: partial derivative of a quantity with respect to a coordinate, e.g. $x$ 
- $\frac{d}{dt}$: total differential, derivative of a quantity with respect to all dependent coordinates ($x, z, y$) 
- $\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla$: Lagrangian (or total) derivative
2.2 Basics - Mathematical add on

- Vector operations
- Nabla operator

Modified slides from Clemens Simmer, University Bonn
2.2 Mathematical add on

Nabla operator – spatial gradient

Nabla - Operator $\vec{\nabla}$:

with $\vec{\nabla} = \nabla = \partial_i \equiv \left( \begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right)$

- Nabla is a (vector) operator that works to the right.
- It only results in a value if it stands left of an arithmetic expression.
- If it stands to the right of an arithmetic expression, it keeps its operator function (and “waits” for application).

Graphically:
- Pooling of the spatial gradients towards the spatial coordinate axes → vector
- Gradient points towards the largest increase of the quantity.
- Amount is the magnitude of the derivative pointing towards the largest increase.
2.2 Mathematical add on

Nabla operator

Product with a Scalar, vector

\[ \vec{\nabla} T \equiv \text{grad } T = \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} T \]

Scalar product "\cdot", scalar

\[ \vec{\nabla} \cdot \vec{u} \equiv \text{div } \vec{u} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \]

Vector product "\times", vector

\[ \vec{\nabla} \times \vec{u} \equiv \text{rot } \vec{u} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix} \]

Unit vectors: 
- \( \vec{i} \): x direction
- \( \vec{j} \): y direction
- \( \vec{k} \): z direction
Nabla operator – algorithms:

Note: Nabla is a (vector) operator, i.e. the sequence must not be changed here!

\[
\begin{pmatrix}
\frac{\partial T}{\partial x} \\
\frac{\partial T}{\partial y} \\
\frac{\partial T}{\partial z}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix} T = \nabla T \neq T \nabla = \begin{pmatrix}
T \frac{\partial}{\partial x} \\
T \frac{\partial}{\partial y} \\
T \frac{\partial}{\partial z}
\end{pmatrix}
\]

\[
\nabla \cdot \vec{u} \equiv \text{div} \vec{u} \equiv \begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix} \cdot \begin{pmatrix}
u \\
w
\end{pmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
\]

\[
\vec{u} \cdot \nabla = \begin{pmatrix}
u \\
w
\end{pmatrix} \cdot \begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}
\]
Forward trajectories started @850 hPa:

- top: between 26.04.-29.04.1986

- bottom: between 30.04.-05.05.1986

Example: nuclear reactor accident Chernobyl on 26.04.1986, 01:30 am

Trajectories (=flight path) are used for the prediction of air movement.

Kraus (2004)
Where is the missing Malaysian Airline MH370 (8 March 2014)?

Particle locations 16 months before reaching La Réunion

Possible locations of MH370 during the last satellite contact

Source: Jonathan Durgadoo, Arne Blaustoch, GEOMAR

GEOMAR press release from 1. September 2015
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3. Equations of motion - nonrotating fluid

I. Laws of motion for a fluid parcel in x-, y-, z-directions (applying Newton's 1. and 2. law)

II. Conservation of mass

III. Law of thermodynamics (including motion)

→ 5 equations for the evolution of the fluid (5 unknowns: \( u, p, T \))
Momentum conservation

The conservation of momentum is represented by Newton’s first law ("Lex prima"):

\[
\text{momentum} = \text{mass} \cdot \text{velocity}
\]

momentum = const. \hspace{1cm} \text{at absence of forces}

Momentum sentence ("Lex secunda"):

temporal change in momentum = Force

\[
M \frac{du}{dt} = F
\]

\( t: \text{time}; M: \text{mass}; u: \text{velocity vector} \ F: \text{Force vector} \)
3. Equations of motion - nonrotating fluid

fluid parcel - cube

Figure 6.2: An elementary fluid parcel, conveniently chosen to be a cube of sides $\delta x$, $\delta y$, $\delta z$, centered on $(x, y, z)$. The parcel is moving with velocity $u$. 

Forces on a fluid parcel

$$\rho \delta x \delta y \delta z \frac{Du}{Dt} = F \quad \text{(Eq. 6-2)}$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

$$= \frac{\partial u}{\partial t} + (u \cdot \nabla) u$$

**ρ**: density

(δx, δy, δz): fluid parcel with infinitesimal dimensions

δM: mass of the parcel; δM = ρ δx δy δz

u: parcel velocity

F: net force

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Gravity

The gravitational force, $g \delta M$, is directed downward:

$$\mathbf{F}_{\text{gravity}} = -g \rho \hat{z} \delta x \delta y \delta z \quad \text{(Eq. 6-3)}$$

$\mathbf{F}_{\text{gravity}}$: Gravitational force
$M$: mass
$g$: gravity acceleration, $g \approx \text{const.}$
$\rho$: density
$\hat{z}$: unit vector in upward direction

3. Equations of motion - nonrotating fluid

**Pressure gradient force**

The *x*-component of the pressure force is:

\[
F(x) = p \cdot A = p \left( x - \frac{\delta x}{2}, y, z \right) \delta y \delta z \\
F(B) = p \cdot B = -p \left( x + \frac{\delta x}{2}, y, z \right) \delta y \delta z
\]

\[
F_x = F(A) + F(B)
\]

Apply Taylor expansion (A.2.1) at midpoint of parcel and neglect small terms (see book):

\[
F_x = -\frac{\partial p}{\partial x} \delta x \delta y \delta z
\]

Apply for all sides (y and z).

---

\( p \): pressure

3. Equations of motion - nonrotating fluid

Pressure gradient force

\[ \mathbf{F}_{\text{pressure}} = (F_x, F_y, F_z) \]

\[ = - \left( \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right) \delta x \delta y \delta z \]

\[ = - \nabla p \delta x \delta y \delta z \quad \text{(Eq. 6-4)} \]

*Figure 6.3: Pressure gradient forces acting on the fluid parcel. The pressure of the surrounding fluid applies a force to the right on face A and to the left on face B.*


\[ p: \text{pressure} \]
Friction force

- Friction force operates on the earth’s surface,
- the greater the surface roughness, the higher the friction force,
- the greater the wind velocity, the higher the friction force,
- friction force takes effect in vertical distances of ~ 100 m up to 1,000 m (→ atmospheric boundary layer Chapter 7).

\[ \mathbf{F}_{\text{friction}} = \rho \mathcal{F} \, \delta x \, \delta y \, \delta z \quad (\text{Eq. 6-5}) \]

\( \mathcal{F} \): frictional force per unit mass (see 7.4.2 and 10.1)
\( \rho \): density
3. Equations of motion - nonrotating fluid

Equations of motion

- All three forces together in Eq. 6-2 gives:

\[ \rho \delta x \delta y \delta z \frac{Du}{Dt} = F_{\text{pressure}} + F_{\text{gravity}} + F_{\text{friction}} \]

- Re-arranging leads to equation of motion for fluid parcels:

\[ \frac{Du}{Dt} + \frac{1}{\rho} \nabla p + g\hat{z} = \mathcal{F} \]

(Equation 6-6)

ρ: density
u: velocity vector
F: Force vector
\( \mathcal{F} \): frictional force per unit mass
p: pressure
g: gravity acceleration
\( \hat{z} \): unit vector in z-direction
Equations of motion – component form

(Eq. 6-6) in Cartesian coordinates:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} & = F_x \quad (6-7a) \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial y} & = F_y \quad (6-7b) \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} + g & = F_z \quad (6-7c)
\end{align*}
\]

---

\( u \): zonal velocity  \\
\( v \): meridional velocity  \\
\( w \): vertical velocity  \\
\( \rho \): density  \\
\( F \): frictional force per unit mass  \\
\( p \): pressure  \\
\( g \): gravity acceleration
Hydrostatic balance

If friction $F_z$ and vertical acceleration $\frac{Dw}{Dt}$ are negligible, we derive from the vertical eq. of motion (6-7c) the hydrostatic balance (Ch.3, Eq.3-3):

$$\frac{\partial p}{\partial z} = - \rho g \quad (Eq. 6-8)$$

Balance between vertical pressure gradient and gravitational force!

“Pressure decreases with height in proportion to the weight of the overlying atmosphere.”

Note: This approximation holds for large-scale atmospheric and oceanic circulation with weak vertical motions.

$\frac{D}{Dt}$: Lagrangian derivative $\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}$
Conservation of mass
– continuity equation

Conserved quantity:
A quantity of which the derivative equals zero.

Conservation of mass for liquids is also called continuity equation.
4. Conservation of mass

Conservation of mass

Figure 6.4: The mass of fluid contained in the fixed volume, $\rho \delta x \delta y \delta z$, can be changed by fluxes of mass out of and into the volume, as marked by the arrows.

Conservation of mass

Mass continuity requires:

\[ \frac{\partial}{\partial t} (\rho \, dx \, dy \, dz) = \frac{\partial \rho}{\partial t} \, dx \, dy \, dz \]

= (net mass flux into the volume)

- mass flux in x-direction per unit time \textit{into} the volume

\[ [\rho u] \left( x - \frac{1}{2} \, dx, y, z \right) \, dy \, dz \]

- mass flux in x-direction per unit time \textit{out of} the volume

\[ [\rho u] \left( x + \frac{1}{2} \, dx, y, z \right) \, dy \, dz \]

4. Conservation of mass

Conservation of mass -2

- net mass flux in x-direction into the volume is then (employing Taylor expansion A.2.2):
  \[- \frac{\partial}{\partial x} (\rho u) \delta x \delta y \delta z ,\]

- net mass flux in y- and z- direction accordingly (see book).

- Net mass flux into the volume:
  \[- \nabla \cdot (\rho u) \delta x \delta y \delta z ,\]

- substituting into mass continuity:
  \[ \frac{\partial}{\partial t} (\rho \delta x \delta y \delta z) = - \nabla \cdot (\rho u) \delta x \delta y \delta z \]

Figure 6.4: The mass of fluid contained in the fixed volume, \( \rho \delta x \delta y \delta z \), can be changed by fluxes of mass out of and into the volume, as marked by the arrows.

Conservation of mass

4. Conservation of mass

leads to equation of continuity:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]  
(Eq. 6-9),

which has the form of conservation law:

\[ \frac{\partial \text{Concentration}}{\partial t} + \nabla \cdot (flux) = 0. \]

Using D/Dt (Eq. 6-1) and \( \nabla \cdot (\rho \mathbf{u}) = \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho \) (see A.2.2) we can rewrite:

\[ \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \]  
(Eq. 6-10)
Approximations for continuity equation: incompressible - compressible flows

- Incompressible flow (e.g. ocean):
  \[ \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (\text{Eq. 6-11}) \]

- Compressible flow (e.g. air, \( \rho \) varies) expressed in pressure coordinate \( p \) applying hydrostatic assumption:
  \[ \nabla_p \cdot \mathbf{u}_p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial p} = 0 \quad (\text{Eq. 6-12}) \]
5. Thermodynamic equation

**Thermodynamic equation**

- Temperature evolution can be derived from first law of thermodynamics (Ch.4, 4-2):

\[
\frac{DQ}{Dt} = c_p \frac{DT}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} \quad \text{(Eq. 6-13)}
\]

- If the heating rate is zero \(\frac{DQ}{Dt} = 0\), Ch. 4.3.1 then:

\[
\frac{DT}{Dt} = \frac{1}{\rho c_p} \frac{Dp}{Dt}
\]

*The temperature of a parcel will decrease/increase in ascent/descent with decreasing/increasing pressure.*

→ Introduction of potential temperature \(\theta\) (Eq. 4-17)

\[
\theta = T \left(\frac{p_0}{p}\right)^\kappa
\]

---

- \(Q\): heat
- \(\frac{DQ}{Dt}\): diabatic heating rate per unit mass
- \(c_p\): specific heat at constant pressure
- \(T\): temperature
- \(\rho\): density
- \(p\): pressure; \(p_0 = 1000\) hPa
- \(\kappa = \frac{R}{c_p}\)
- \(R\): gas constant for dry air

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Thermodynamic equation - potential temperature

- Eq. 6-13 for potential temperature $\theta$ *(see book for details)*:

$$\frac{D\theta}{Dt} = \left(\frac{p}{p_0}\right)^{-\kappa} \frac{DQ}{c_p} \quad (\text{Eq. 6-14})$$

Note: For adiabatic motions ($\delta Q = 0$) $\theta$ is conserved.
Summary

→5 equations for the evolution of the fluid (5 unknowns)

Ia.) \[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = F_x \] (6-7a)

Ib.) \[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial y} = F_y \] (6–7b)

Ic.) \[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} + g = F_z \] (6-7c)

II.) \[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \] (6-9 or 6-11/6-12)

III.) \[ \frac{DQ}{Dt} = c_p \frac{DT}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} \] (6-13 or 6-14)

Restrictions:
- application to average motion is often incorrect i.e. turbulence,
- fixed coordinate system.
2.2 Basics - Mathematical add on

- Vector operations
- Nabla operator

Modified slides from Clemens Simmer, University Bonn
Vector operations
- Multiplication with a scalar $a$

$$a\vec{v} = \vec{v}a = av_i = v_ia = a\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} av_x \\ av_y \\ av_z \end{pmatrix}$$

- **With a scalar multiplication the vector stays a vector.**
- Each element of a vector is multiplied individually with a scalar $a$.
- The vector extends (or shortens) itself by the factor $a$.
- **Convention:** *With the multiplication scalar – vector we don’t use a point (like with scalar – scalar).*
Vector operations
- Scalar product -

\[
\vec{v} \cdot \vec{f} = \vec{f} \cdot \vec{v} = \left( \begin{array}{c} v_x \\ v_y \\ v_z \end{array} \right) \cdot \left( \begin{array}{c} f_x \\ f_y \\ f_z \end{array} \right) = v_x f_x + v_y f_y + v_z f_z = f_i v_i \equiv \sum_i f_i v_i = f_i v_i
\]

- The scalar product of two vectors is a scalar.
- It is multiplied component-wise, then added.
- Convention: *The scalar product is marked by a multiplication sign (·).*
- It is at a maximum with parallel vectors and disappears when the vectors are perpendicular to each other.
Vector operations – vector product

\[ \mathbf{c} = \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \mathbf{i} (a_y b_z - a_z b_y) + \mathbf{j} (a_z b_x - a_x b_z) + \mathbf{k} (a_x b_y - a_y b_x) \]

\[ \mathbf{c} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix} \]

\[ |\mathbf{c}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \alpha, \quad \mathbf{c} \perp \mathbf{a} \wedge \mathbf{c} \perp \mathbf{b} \]

- The vector product of two vectors is again a vector.
- Convention: The vector product or cross product is marked by an “x”.
- If \( \mathbf{a} \) is turned to \( \mathbf{b} \) on the fastest route possible, the vector (from the vector product) points in the direction in which a right-hand screw would move (right hand rule).
- With parallel vectors, it disappears, and it reaches its maximum when the two vectors are at right angles to each other!
Forces on rotating sphere

Fictitious forces:

- occur in revolving/accelerating system
  (e.g. car in curve)
- occur on the earth’s surface
  (Fixed system on earth’s surface forms an accelerated system, because a circular motion is performed once a day. Movement of earth around the sun compared to earth’s rotation is insignificant.)

→ Coriolis force

→ Centrifugal force
Coordinate systems in meteorology

- Cartesian coordinates \((x, y, z, t)\)
- Spherical coordinates \((\lambda, \varphi, z, t)\)
- Height coordinates: geometrical altitude \((z)\), pressure \((p)\) and potential temperature \((\theta)\)

3. Equations of motion - nonrotating fluid

\[ \text{Figure 6.19: At latitude } \varphi \text{ and longitude } \lambda, \text{ we define a local coordinate system such that the three coordinates in the } (x, y, z) \text{ directions point (eastward, northward, upward): } \begin{align*} dx &= a \cos \varphi \, d\lambda; \quad dy = ad\varphi; \quad dz = dz, \text{ where } a \text{ is the radius of the Earth. The velocity is } u = (u, v, w) \text{ in the directions } (x, y, z). \text{ See also Appendix A.2.3.} \]
Inertial - rotating frames

Figure 6.9: On the left is the velocity vector of a particle $u_{in}$ in the inertial frame. On the right is the view from the rotating frame. The particle has velocity $u_{rot}$ in the rotating frame. The relation between $u_{in}$ and $u_{rot}$ is $u_{in} = u_{rot} + \Omega \times r$, where $\Omega \times r$ is the velocity of a particle fixed (not moving) in the rotating frame at position vector $r$. The relationship between the rate of change of any vector $A$ in the rotating frame and the change of $A$ as seen in the inertial frame is given by: $(DA/DT)_{in} = (DA/DT)_{rot} + \Omega \times A$.

$r$ : position vector; $\Omega$ (Greek: omega): rotation vector
Chapter 6: The equations of fluid motion

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Ch. 6 - The equations of fluid motions

1. Motivation

2. Basics
   2.1 Lagrange - Euler
   2.2* Mathematical add on

3. Equation of motion for a nonrotating fluid
   3.1 Forces
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   3.3 Hydrostatic balance

4. Conservation of mass

5. Thermodynamic equation

6. Equation of motion for a rotating fluid
   6.1 Forces
   6.2 Equations of motion

7. Take home messages

*Addition, not in book.
Recap: 5. Equations of motion - rotating fluid

Inertial - rotating frames

Figure 6.9: On the left is the velocity vector of a particle $u_{in}$ in the inertial frame. On the right is the view from the rotating frame. The particle has velocity $u_{rot}$ in the rotating frame. The relation between $u_{in}$ and $u_{rot}$ is $u_{in} = u_{rot} + \Omega \times r$, where $\Omega \times r$ is the velocity of a particle fixed (not moving) in the rotating frame at position vector $r$. The relationship between the rate of change of any vector $A$ in the rotating frame and the change of $A$ as seen in the inertial frame is given by: $(DA/Dt)_{in} = (DA/Dt)_{rot} + \Omega \times A$.

$r$ : position vector; $\Omega$ (Greek: omega): rotation vector, $A$: any vector

Transformation into rotating coordinates

Consider figure 6.9:

\[ \mathbf{u}_{\text{in}} = \mathbf{u}_{\text{rot}} + \mathbf{\Omega} \times \mathbf{r} \]  
(Eq. 6-24)

\[ \frac{D}{Dt} \mathbf{A}_{\text{in}} = \left( \frac{D}{Dt} \right)_{\text{rot}} \mathbf{A}_{\text{rot}} + \mathbf{\Omega} \times \mathbf{A} \]  
(Eq. 6-26)

We set \( \mathbf{A} = r \) and \( \mathbf{A} \rightarrow \mathbf{u}_{\text{in}} \) in Eq. 6-26 using 6-24, we derive:

\[
\left( \frac{D\mathbf{u}_{\text{in}}}{Dt} \right)_{\text{in}} = \left[ \left( \frac{D}{Dt} \right)_{\text{rot}} + \mathbf{\Omega} \times \right] \left( \mathbf{u}_{\text{rot}} + \mathbf{\Omega} \times \mathbf{r} \right)
\]

\[
= \left( \frac{D\mathbf{u}_{\text{rot}}}{Dt} \right)_{\text{rot}} + 2\mathbf{\Omega} \times \mathbf{u}_{\text{rot}} + \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}
\]  
(Eq. 6-27)

\( \mathbf{r} \): position vector
\( \mathbf{A} \): any vector
\( \mathbf{\Omega} \): rotation vector
\( \left( \frac{D\mathbf{r}}{Dt} \right)_{\text{rot}} = \mathbf{u}_{\text{rot}} \)
Rotating equations of motion

Substituting \( \left( \frac{Du_{in}}{Dt} \right)_{in} \) from Eq. 6-27 into Eq. 6-6 (inertial frame equation of motion) we derive in rotating frame, dropping subscript “rot“:

\[
\frac{Du}{Dt} + \frac{1}{\rho} \nabla p + g \hat{z} = F \quad \text{(Eq. 6-6)}
\]

\[
\frac{Du}{Dt} + \frac{1}{\rho} \nabla p + g \hat{z} = -2\Omega \times u - \Omega \times \Omega \times r + F \quad \text{(Eq. 6-28)}
\]

\( u \): velocity vector, \( t \): time, \( p \): pressure, \( g \): gravity acceleration, \( \hat{z} \): unit vector in z-direction

\( \Omega \): rotation vector, \( F \): frictional force per unit mass
Centrifugal force – Centripetal force

- Directed radially outward.
- If no other forces are present the particle would accelerate outwards.
- Fictitious force, balanced by Centripetal force

\[
F_C = -M \, \Omega \times (\Omega \times r)
\]

\[
F_{Cp} = - F_C
\]
5. Equations of motion - rotating fluid

### Centrifugal acceleration – modified gravitational potential

Combine the gradient of Centrifugal potential $-\Omega \times \Omega \times r = \nabla \left( \frac{\Omega^2 r^2}{2} \right)$ and Gravitational potential $g\hat{z} = \nabla (gz)$ in Eq. 6-28:

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho} \nabla p + \nabla \Phi = -2\Omega \times \mathbf{u} + \mathbf{F}$$  \hspace{1cm} \text{Eq. 6–29}

- Pressure gradient
- Gravitational + Centrifugal accelerations
- Coriolis acceleration
- Friction acceleration

• $\Phi = gz - \frac{\Omega^2 r^2}{2}$  \hspace{1cm} \text{Eq. 6–30}

$\Phi$ (Greek “Phi”) is modified gravitational potential on Earth.
On the sphere

Shallow atmosphere approx. \( a+z \approx a \):
\[
r = (a+z)\cos\varphi \approx a \cos\varphi,
\]
Eq. 6-30 becomes:
\[
\Phi = gz - \frac{\Omega^2 a^2 \cos^2\varphi}{2}
\]

Definition of geopotential surfaces:
\[
z^* = z + \frac{\Omega^2 a^2 \cos^2\varphi}{2g} \quad \text{(Eq. 6-40)}
\]

\( \Omega = 7.27 \times 10^{-5} \text{ s}^{-1} \)
\( a = 6.37 \times 10^6 \text{ m} \)
\[
\rightarrow \text{At Equator: } \frac{\Omega^2 a^2}{2g} \approx 11 \text{ km}
\]

5. Equations of motion - rotating fluid

Coriolis acceleration

\[ \frac{Du}{Dt} = -2\Omega \times u \quad \text{Eq. 6-31} \]

NH (viewed from above the Northpole): 
\( \Omega > 0 \): rotation anticlockwise

SH (viewed from above the Southpole): 
\( \Omega < 0 \): rotation clockwise

Absence of other forces:

\[ \Omega = \begin{pmatrix} 0 \\ \Omega_y \\ \Omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ \Omega \cos \varphi \\ \Omega \sin \varphi \end{pmatrix} : \text{Angular velocity} \]

5. Equations of motion - rotating fluid

**Coriolis force on the sphere – Coriolis parameter**

Assuming that 1) $\Omega_z$ is negligible as $\Omega u \ll g$; 2) $w \ll u_h$ leads to:

$$-2\Omega \times u \approx (-2\Omega \sin \varphi v, 2\Omega \sin \varphi u, 0)$$

(Eq. 6-41)

$$= f \hat{z} \times u$$

**Coriolis parameter $f$:**

$$f = 2\Omega \sin \varphi$$

(Eq. 6-42)

Note: Vertical component of the Earth's rotation rate, which matters.

| TABLE 6.1. Values of the Coriolis parameter, $f = 2\Omega \sin \varphi$ (Eq. 6-42), and its meridional gradient, $\beta = df/dy = 2\Omega/a \cos \varphi$ (Eq. 10-10), tabulated as a function of latitude. Here $\Omega$ is the rotation rate of the Earth and $a$ is the radius of the Earth. |
|-----------------|-----------------|-----------------|
| Latitude | $f \left( \times 10^{-4} \text{s}^{-1} \right)$ | $\beta \left( \times 10^{-11} \text{s}^{-1} \text{m}^{-1} \right)$ |
| 90°   | 1.46            | 0               |
| 60°   | 1.26            | 1.14            |
| 45°   | 1.03            | 1.61            |
| 30°   | 0.73            | 1.98            |
| 10°   | 0.25            | 2.25            |
| 0°    | 0               | 2.28            |

$\Omega = 7.27 \times 10^{-5} \text{s}^{-1}$

$a = 6.37 \times 10^6 \text{m}$

Coriolis force

In direction of movement, in

**NH:** deflection of air particles to the right
takes place.

**SH:** deflection of air particles to the left

directional movement
Equations of motion — Coriolis parameter

\[
\frac{Du}{Dt} + \frac{1}{\rho} \nabla p + \nabla \Phi + f \hat{z} \times u = F \quad \text{(Eq. 6–43)}
\]

- Fluid in a thin spherical shell on a rotating sphere, applying hydrostatic balance for vertical component and neglecting \(F_z\) compared with gravity:

\[
\frac{Du}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial x} - f v = F_x \quad \text{(Eq. 6–44)}
\]

\[
\frac{Dv}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial y} + f u = F_y
\]

\[
\frac{1}{\rho} \frac{\partial p}{\partial z} + g = 0
\]
• 5 equations for the evolution of the fluid with 5 unknowns \((u,v,w,p,T)\): 
  equations of motion (3), conservation of mass (1), thermodynamic equation (1)

• Equations of motion on a non-rotating fluid: 
  Pressure gradient force, gravitational force and friction force act. 
  \[
  \frac{D\mathbf{u}}{Dt} + \frac{1}{\rho} \nabla p + g\hat{z} = \mathbf{F}
  \]

• Equations of motion on a rotating fluid: 
  Pressure gradient force, modified gravitational potential (gravitational and centrifugal force), Coriolis force and friction force act. 
  \[
  \frac{D\mathbf{u}}{Dt} + \frac{1}{\rho} \nabla p + \nabla \Phi + f\hat{z} \times \mathbf{u} = \mathbf{F}
  \]