Problem 1: Equation of state

Water density is a function of temperature, salinity and pressure

$$\rho = \rho(T, S, p).$$

a. (2 points) How does density change with each of the independent variables? Explain why temperature is really not a completely independent variable.

b. (2 points) What is *potential density* and why do we use this concept rather than the full density when investigating the static (vertical) stability of a water column?

Problem 2: Salt conservation equation

The equation for conservation of salt, written in terms of salinity (salt per unit volume), is

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\nabla \cdot (\boldsymbol{v} S) + \kappa_S \nabla^2 S \\ &= -\left[\frac{\partial (uS)}{\partial x} + \frac{\partial (vS)}{\partial y} + \frac{\partial (wS)}{\partial z}\right] + \kappa_S \left[\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}\right]\end{aligned}$$

a. (2 points) Explain what the different terms mean (you can choose either the top or bottom description).

b. (2 points) Explain why, under the Boussinesq approximation, the equation can also be written

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\boldsymbol{v} \cdot \nabla S + \kappa_S \nabla^2 S \\ &= -\left[u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z}\right] + \kappa_S \left[\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}\right].\end{aligned}$$

c. (2 points) Rewrite the equation for a water parcel which follows the flow (i.e. in the Lagrangian rather than the Eulerian description).

Problem 3: Geostrophy and thermal wind

Under the so-called primitive equations the horizontal momentum equations are

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y},\end{aligned}$$

where we have ignored the effects of friction.

a. (2 points) Scale one of these equations and explain the conditions (introduce the temporal and advective *Rossby numbers*) under which we have a balance between the Coriolis accelleration and the horizontal pressure gradient, i.e. what we call *the geostrophic balance.*

b. (3 points) Figure 1 illustrates that geostrophic currents in *unstratified* rotating fluids don't change with depth. So if bottom currents have to circumnavigate an obstacle, so do the surface currents. Why is this? (Hint: in an unstratified fluid the horizontal pressure gradients are entirely due to the sea surface tilt.)

c. (2 points) When the ocean is stratified geostrophic currents *can* change with depth and the depth changes are related to horizontal density gradients. The expressions are

$$\frac{\partial u}{\partial z} = \frac{g}{f\rho_0}\frac{\partial \rho}{\partial y}$$
$$\frac{\partial v}{\partial z} = -\frac{g}{f\rho_0}\frac{\partial \rho}{\partial x}$$

Using the geostrophic momentum equation and the hydrostatic balance,

$$\frac{\partial p}{\partial z} = -\rho g,$$

show why this is the case.

d. (2 points) At the sea surface ($z = 0$) we write the east-west pressure gradient as

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} = -g\frac{\partial \eta}{\partial x},$$

where η is the sea surface height above $z = 0$. In an ocean region around 43°N the sea surface tilts upward towards the east by 0.5 m over 100 km. Assuming Coriolis parameter $f = 10^{-4}\mathrm{s}^{-1}$ and gravitational accelleration $g = 10\,\mathrm{m\,s^{-2}}$, estimate the meridional surface geostrophic velocity. Is it pointing northward or southward?

e. (3 points) Below the surface, in the same ocean region, the density decreases eastward, with $\partial\rho/\partial x = -10^{-5}\mathrm{kg\,m^{-3}m^{-1}}$. Assume that this horizontal density gradient doesn't change with depth and then estimate the depth at which the meridional geostrophic velocity is zero. You can assume a reference density $\rho_0 = 1000\,\mathrm{kg\,m^{-3}}$. (If you don't have the answer to the question above, just set up the expressions.)

Problem 4: Ekman pumping and wind-driven large-scale flows

a. (2 points) Figure 2 shows the surface Chlorophyl concentration in the eastern tropical Pacific. Enhanced values (green, yellow and red) indicate

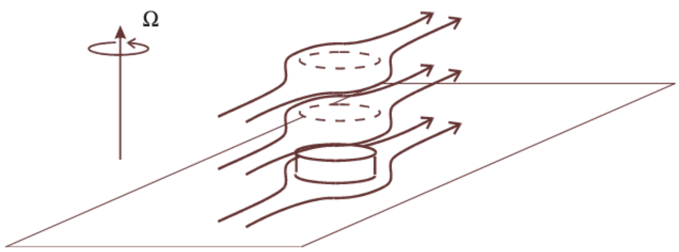

Figure 1: Geostrophic flows in unstratified fluids that tend to flow around bottom obstacles. And the surface currents also flow around (not over) the obstacle.

that sub-surface water has reached the surface. Explain the mechanism leading to this. What direction do the winds need to have along the equator to create the observed pattern? What about the direction along the Peruvian coast?

b. (2 points) The depth-integrated surface Ekman transport is given by

$$
\begin{aligned}
V_E &\equiv \int_{z_0}^{0} v_a dz = -\frac{\tau_x^w}{f\rho_0} \\
U_E &\equiv \int_{z_0}^{0} u_a dz = \frac{\tau_y^w}{f\rho_0},
\end{aligned}
$$

where τ_x^w and τ_y^w are the wind stress components in the zonal and meridional directions, respectively. Show that in the open ocean the vertical velocity at the base of the Ekman layer is

$$
\begin{aligned}
w(z_0) &= \left[\frac{\partial}{\partial x}\left(\frac{\tau_y^w}{\rho_0 f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_x^w}{\rho_0 f}\right)\right] \\
&= \nabla \times \left(\frac{\boldsymbol{\tau}^w}{\rho_0 f}\right).
\end{aligned}
$$

c. (2 points) The Sverdrup and Stommel equations relating the depth-integrated meridional flow to either the curl of the wind stress or to the curl of the bottom velocity of the ocean is

$$
\begin{aligned}
\beta V &= \frac{1}{\rho_0}\left(\frac{\partial \tau_y^w}{\partial x} - \frac{\partial \tau_x^w}{\partial y}\right) - R\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\
&= \frac{1}{\rho_0}\left(\nabla \times \boldsymbol{\tau}^w\right) - R\nabla \times \boldsymbol{u},
\end{aligned}
$$

where R is a bottom friction parameter (with units $\mathrm{m\,s^{-1}}$). The flow in the ocean interior, away form continental boundaries, is assumed to be in a so-called *Sverdrup balance*. In terms of the above equation, what is this balance?

d. (3 points) The Sverdrup balance cannot hold over the entire ocean if two east-west boundaries are present. Why not? Figure 3 shows a southward Sverdrup transport with return flow in frictional boundary layers either in the east or in the west. Explain why the western boundary layer is in fact the only possible one.

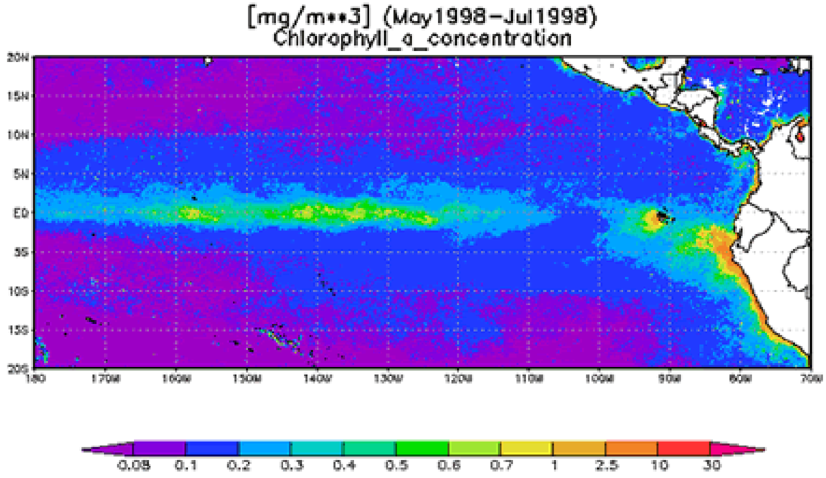

Figure 2: Surface chlorophyl concentrations in the eastern tropical Pacific Ocean.

Problem 5: Equatorial dynamics

a. (1 point) Why do our theories of the large-scale wind-driven ocean circulation developed for mid-latitudes fail at the equator?

b. (3 points) Figure 4 shows a vertical cross-section of zonal (east-west) currents in the equatorial Pacific Ocean. At the equator itself is the *Equatorial Undercurrent*, with peak velocities at around 150 m depth. Explain the dynamics of this current.

Problem 6: High-frequency (non-rotational) surface waves

The dispersion relation for a high-frequency surface gravity wave, when Earth's rotation can be neglected, is

$$\omega^2 = gk \tanh(kH),$$

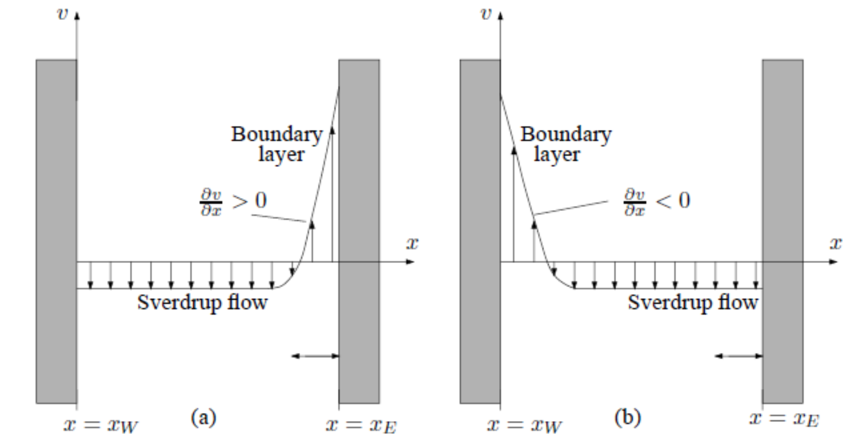

Figure 3: Southward Sverdrup transport with frictional (Stommel) boundary layers either in the east or in the west.

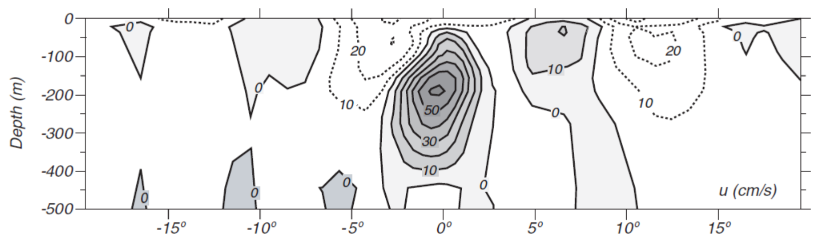

Figure 4: Vertical cross section of zonal currents in the equatorial Pacific Ocean.

where g is the gravitational accelleration, H is the ocean depth and k is the wavenumber (we assume a wave travelling in the x-direction).

a. (2 points) From the complete expression, derive the deep-water and shallow-water limits

$$\begin{aligned} \omega &= \sqrt{gk} \qquad && H/\lambda \gg 1 \\ \omega &= k\sqrt{gH} \qquad && H/\lambda \ll 1, \end{aligned}$$

where $\lambda = 2\pi/k$ is the wavelength.

b. (2 points) From these last two expressions, find the corresponding phase velocities. Explain why deep-water waves are called *dispersive* waves whereas shallow-water waves are not.

c. (2 points) Explain the difference between *wind sea* and *swell*.

d. (2 points) When the swell reaches the coast it tends to do so with the wave crests always parallell to the beach. Explain the process leading to this result.

Problem 7: The tides

a. (3 points) Explain a) the tide-generating force and b) the equilibrium tide, considering only the effect of the moon (ignoring the sun). Using one or two drawings would be helpful.

b. (2 points) Explain why we have semi-diurnal (twice daily) and sometimes also diurnal (once daily) equilibrium tides.

c. (2 points) We don't actually observe the equilibrium tide but rather travelling waves, so-called Poincaré or Kelvin waves. The dispersion relation for Poincaré waves is

$$\omega = \pm\sqrt{f^2 + gHk^2}$$

or, if we divide by the Coriolis parameter,

$$\frac{\omega}{f} = \pm\sqrt{1 + (L_d k)^2},$$

where $L_d = \sqrt{gH}/f$. This dispersion relationship is shown in Figure 5. What is the parameter L_d called and what is its meaning?

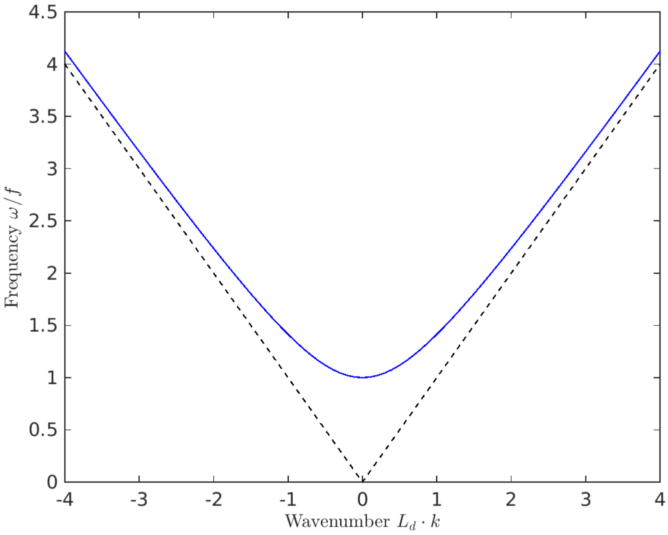

Figure 5: The dispersion relation for Poincaré waves traveling in the x-direction (blue solid line) and for high-frequency shallow-water waves for which Earth's rotation is not important (black dashed line).