Computer problems

connected to lecture notes on:

Atmospheres and oceans on computers

Lars Petter Røed and Thor Erik Nordeng
Norwegian Meteorological Institute
P. O. Box 43 Blindern
0313 Oslo, Norway
and
Department of Geosciences
University of Oslo
Section Meteorology and Oceanography
P. O. Box 1022 Blindern
0315 Oslo, Norway

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PREFACE

Presented are exercises connected to the lecture notes on “Fundamental of Atmospheres and Oceans on Computers”. A certain number of these are obligatory and must be submitted for evaluation. The students may themselves decide which six problems to submit. The exercises replace the middle week exam, and must be approved before the student is allowed to take the (oral) exam.

I strongly encourage and recommend those students who reads the lecture notes also to go through the exercises contained herin. The main reason being that to solve atmospheric and oceanographic problems using numerical methods consist of three stages. The first is to develop what one think is a reasonable numerical algorithm that can be used to replace the underlying continuous, mathematical problem. The scond is to program a code for this algorithm on a given computer so that the computer calculates the results. This stage also includes what is called debugging and verification, that is, checking that the code is a true replication of the numerical algorithm and actually is doing what one think one has programmed. The final and third stage is to visualize the results in a reasonable way and to discuss the results. While the lectures notes gives some insight into the first stage and may be gives some guidelines into the second stage, these exercises gives the practical hands-on experience necessary to get some insight into the two latter stages.

The exercises will be continuously amended with more exercises to adjust to the new lecture notes. Some of the exercises are based on exercises made for earlier edition of the lecture notes, and the authors would like to thank the many colleagues who has contributed to develop these exercises over the years, and to the many students for pointing out misprints and other mistakes. Good luck!

Blindern, September 19, 2007
Lars Petter Røed (sign.)

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Computer problem 1:
Truncation error in a recursion formulae with two terms

a.

Let

$$\pi = 4\arctan(1),\ Z_1 = \pi, \text{ og } S_1 = \pi. \tag{1}$$

Compute

$$Z_{i+1} = 3.1Z_i - 2.1Z_1 \text{ og } S_{i+1} = \left(\frac{9.}{5.}\right) S_i - \left(\frac{4.}{5.}\right) S_1 \tag{2}$$

for $i = 1(1)100$. Compute also the *relative* error (in percent) for each i. Write π, Z_i, S_i and the realtive error in percent. The output should be readable and easy to understand in itself and should have headings for each column. Enclose the program code. Do the problem on different platform (from handhelds to PC's and supercomputers) if possible and available to you. Experiment by using differnt contants in the recursion formulae. Does it make a difference in the answer?

The purpose of the exercise is twofold: 1) It is simple enough to enable you to refresh your knowledge of FORTRAN, and (although you don't have to do the exercise using FORTRAN) without having to write lengthy codes, and 2) it demonstrates the dramatic consequences of even small or insignificant truncation errors (which is always there in numerical computations).

b.

Show analytically why the recursion formulas for Z_i og S_i do not compute π correctly.

Computer problem 2:
Diffusive processes in the ocean and atmosphere

In the ocean and atmosphere the vertical (and horizontal) heat exchange is a dominantly a turbulent process. Commonly this turbulent heat exchange is parameterized as a diffusion process, or down the gradient parameterization. Thus in its simplest and purest form the vertical heat exchange is governed by a diffusion equation, that is,

$$\partial_t \theta = \nabla \cdot (\kappa \nabla \theta) \tag{3}$$

where θ is the potential temperature and κ is the diffusion coefficient, t is time, and ∇ is the three-dimensional del-operator, that is,

$$\nabla \theta = \mathbf{i}\partial_x \theta + \mathbf{j}\partial_y \theta + \mathbf{k}\partial_z \theta. \tag{4}$$

where x, y are the two horizontal axes and z is the vertical coordinate in a geopotential coordinate system. Since the heat exchange is turbulent the diffusion coefficient is a function of space and time, and thus is inside the del-operator. Mathematically we say that θ is the dependent variable, while x, y, z, t are the independent variables. In fact θ can be any active (like temperature, humidity and salinity) or passive tracer (like CO_2).

In this exercise we will simplify the diffusion to a vertical process only, that is, $\theta = \theta(z, t)$. Furthemore, we let the diffusion coefficient be a constant. Under these circumstances (3) reduces to the one-dimensional, diffusion equation, that is,

$$\partial_t \theta = \kappa \partial_z^2 \theta. \tag{5}$$

a.

Develop a numerical scheme (or finite difference analog) that is forward in time and centered in space for (5). There are two boundaries, one at the bottom and one at the top. We will assume that the bottom is located at $z = 0$ and that the top is located at $z = D$. At the bottom and top boundaries we will assume that the temperature is fixed at the freezing point, that is, $\theta(z = 0, t) = \theta(z = D, t) = 0^o C$. Initially the temperature is a certain function of depth z, say $\theta(z, t = 0) = \Theta(z)$.

b.

Show that the forward in time, centered in space scheme is stable under the condition $K \leq \frac{1}{2}$, where

$$K = \frac{\kappa \Delta t}{\Delta z^2}, \tag{6}$$

Δt is the times step and Δz is the distance between two consequtive discrete depths (commonly referred to as the mesh or grid size).

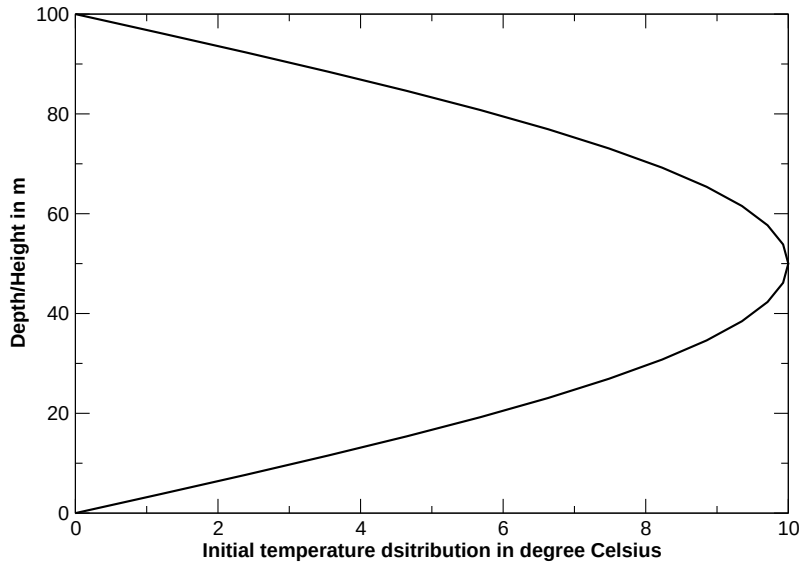

Figure 1: Initial temperature distribution according to (7).

c.

Find the numerical solution to (5) using the above scheme for $t = 0, N\Delta t$ where $N = 200$ and Δt the time step ($t = n\Delta t, n = 0(1)N$). Let

$$\Theta(z) = \Theta_0 \sin(\frac{\pi z}{D}), \quad z \in [0, D], \tag{7}$$

as shown in Figure 1, $D = 100$ m, $\Delta z = D/26$, and $\Theta_0 = 10^o C$. $n = 0(1)200$ where n is the time step counter ($t = n\Delta t$). Do this twice once with $K = 0.45$ and next with $K = 0.55$. Plot the results for $n = 0$, $n = 100$ og $n = 200$.

d.

Assess and discuss the solutions.

e.

Develop a finite difference analog or algorithm for (5) which is centered in both time and space and show that this algorithm is unconditionally unstable in a numeric sense, i.e., unstable for any choice made for Δt and Δz.

Computer problem 3:
Advection in atmosphere and oceans

In some cases an environmental problem in one location has its origin in quite another location. For instance an emission of sulfurous substances in one location is transported with the atmospheric circulation and deposited in another location. Other oceanic examples are transportation of nutrients and fish larvae, oil drift, and drifting objects of any kind. Other common examples in the atmosphere and ocean are transportation and spreading of radionuclide's and heavy metals and other contaminants and/or chemical substances. Commonly these substances are tracers just like, e.g., temperature and their transportation and spreading is governed by an advection-diffusion equation, say

$$\partial_t \theta + \nabla \cdot (\mathbf{v}\theta) = \nabla \cdot (\kappa \nabla \theta) \tag{8}$$

where θ is the concentration of the tracer, $\mathbf{v}$ is the three-dimensional wind or current vector and κ is the diffusion coefficient. While, as shown in Computer problem 2, the diffusive part (the term on the right-hand side of eq. 8) acts to spread the tracer by smoothing any differences, the advective part (second term on the left-hand side of eq. 8) acts to transport or advect the tracer concentration form one location to the other. This is essentially the background for the present computer problem.

We will first simplify the problem and consider a pure advective process only. The problem is then reduced to

$$\partial_t \theta + \nabla \cdot (\mathbf{v}\theta) = 0. \tag{9}$$

To simplify further we will also assume that the advection is only in the direction of the horizontal x-axis and that the velocity is constant. Thus $\mathbf{v} = u\mathbf{i}$ where u is independent of time and space. Our task is thus to solve the one-dimensional advection equation

$$\partial_t \theta + u \partial_x \theta = 0 \quad \text{for } x \in < 0, L > \tag{10}$$

with appropriate boundary and initial conditions, one initial condition for $t = 0$ and one condition in space, say at $x = 0$.

To this end we will make use of three schemes, namely the leapfrog scheme,

$$\frac{\theta_j^{n+1} - \theta_j^{n-1}}{2\Delta t} + u \frac{\theta_{j+1}^n - \theta_{j-1}^n}{2\Delta x} = 0, \tag{11}$$

the upwind scheme

$$\frac{\theta_j^{n+1} - \theta_j^n}{\Delta t} + \frac{u}{\Delta x} \begin{cases} \theta_{j+1}^n - \theta_j^n & \text{if } u < 0 \\ \theta_j^n - \theta_{j-1}^n & \text{if } u \geq 0 \end{cases} = 0, \tag{12}$$

and the diffusive scheme

$$\frac{\theta_j^{n+1} - \frac{\theta_{j+1}^n + \theta_{j-1}^n}{2}}{\Delta t} + u \frac{\theta_{j+1}^n - \theta_{j-1}^n}{2\Delta x} = 0. \tag{13}$$

for $j = 1(1)J$ where $x = 0$ is associated with $j = 1$ and $x = L$ with $j = J$. In contrast to the leapfrog scheme, which is centered in time and space, the upwind scheme is a simple forward in time and forward in space scheme. The diffusive scheme is special. It is centered in space and forward in time, except that θ_j^n is replaced by $\frac{1}{2}(\theta_{j+1}^n + \theta_{j-1}^n)$.

a.

Show that all schemes are numerically stable under the condition

$$C = \frac{|u|\Delta t}{\Delta x} \leq 1 \tag{14}$$

where C is the Courant number.

b.

Show that a scheme which solves (10) employing a forward in time and centered in space scheme is unconditionally unstable for all choices of Δt and Δx.

c.

Show that the upwind scheme (12) inherently includes a numerical diffusion with a diffusion coefficients given by

$$\frac{1}{2}|u|\Delta x(1 - C) \tag{15}$$

where C is still the Courant number. Show also that the the diffusive scheme (13) likewise introduces a numerical diffusion with a coefficient given by

$$\frac{\Delta x^2}{2\Delta t}(1 - C^2). \tag{16}$$

In the following we will make use of periodic boundary conditions, that is, we will require that $\theta(x, t) = \theta(x + L, t)$. Numerically this implies that $\theta_1^n = \theta_J^n$, $\theta_2^n = \theta_{J+1}^n$, and so forth. Furthermore, we will make two experiments who differs only in the specification of the initial condition. The first is a simple sinusoidal distribution,

$$\theta_j^0 = \Theta_0 \sin\left(\frac{2\pi x_j}{L}\right) \quad j = 1(1)J, \tag{17}$$

where Θ_0 is a reference value. The second is

$$\theta_j^0 = \Theta_0 \begin{cases} 0.00 & \text{for } j \leq 48 \\ 0.75 & \text{for } j = 49 \\ 1.00 & \text{for } j = 50 \\ 0.75 & \text{for } j = 51 \\ 0.00 & \text{for } j \geq 52 \end{cases}, \tag{18}$$

and is constructed in order to display the peculiarities of the various schemes in the presence of sharp gradients.

d.

Replace (10) by its dimensionless counterpart, and solve this using the three numerical algorithms (or schemes). In this we will make use of the the two different initial conditions (17) and (18), and the periodic boundary condition. Furthermore, let $x_j = (j-1)\Delta x$ and $J = 101$. Do one experiment with $C = 0.5$ and another with $C = 1$. Do also other experiments in which the Courant number takes on values between $\frac{1}{2}$ and 1. Let the time step counter n start at 0 and stop the calculation when $n = 200$. The increment of n should be 1, thus $n = 0(1)200$). Plot the solution for each of the two initial conditions, that is, six graphs total for $n = 0$, $n = 30$ and $n = 100$. Furthermore, plot the solution for $n = 0$ and $n = 200$ in the same graph using (18) as initial condition for each of the three schemes.

e.

Discuss the solution based on the plots. How does the solution develop in time? Which of the solutions are dissipative and which are dispersive? What is the characteristics of these latter processes?

Computer problem 4:
Yoshida's equatorial jet current

We consider an "infinite" equatorial ocewan consisting of two immiscible layers of with a density difference $\Delta\rho$ (Figure 2). The lower layer, with a density given by the reference density ρ_0, is thick with respect to the upper layer. At time $t = 0$ the ocean is at rest, at which time the thickness of the upper layer equals its equilibrium depth H. At this particular time the ocean is forced into motion by turning on a westerly wind (wind from the west).

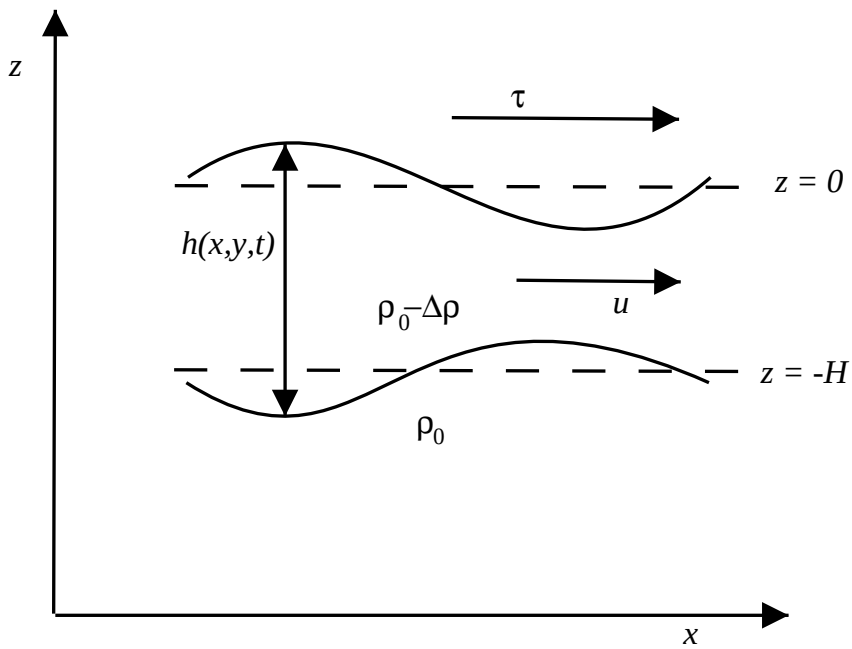

Figure 2: Sketch of a reduced gravity ocean model consisting of two layers with a density difference given by $\Delta\rho$.

The governing equations of such a "reduced gravity" model of the ocean, is

$$\partial_t u - \beta y v = \frac{\tau^x}{\rho_0 H} \tag{19}$$

$$\partial_t v + \beta y u = -g' \partial_y h \tag{20}$$

$$\partial_t h + H \partial_y v = 0 \tag{21}$$

Here u, v are the respectively the eastbound and westbound components of the velocity in a Cartesian coordinate system (x, y, z) with x directed eastward along the equator, y directed northwards with $y = 0$at the equator, and z directed along the negative gravitational force, that is upwards (Figure 2). The impact of the Earth's rotation is given by the Coriolis parameter $f = 2\Omega \sin\phi$ where Ω is the Earth's rotation rate and ϕ is the latitude. The westerly wind

is given by the wind stress component τ^x which is fixed in time. Furthermore, we define the reduced gravity by $g' \equiv g(\Delta\rho_0/\rho)$ where g is the gravitational acceleration. The instantanuous thickness of the upper layer is given by h. The notation ∂_t and ∂_y is used to denote differentiation with respect to t and y respectively.

Note that at the equator $f = 0$ and that it increases with increasing latitude. A simplified parameterization of this effect is through the so called β-plane approximation,

$$f = \beta y, \quad \text{hvor} \quad \beta = \partial_y f|_{y=0}. \tag{22}$$

We note the β is just a measure of the first term in a Taylor series of f at the equator. Thus it represents the effect to first order of the impact of the change in the Earth's rotation rate with latitude.

a.

Show that the inertial oscialltions (that is oscialltion with a frequency equal the inertial frequency f) is eliminated by neglecting $\partial_t v$ in (20).

b.

Show that the problem, given that the inertial oscillation are eliminated, is reduced to the ordinary differential equation

$$L^4 \partial_y^2 v - y^2 v = aLy \tag{23}$$

where

$$L = \sqrt{\frac{c}{\beta}}, \quad a = \frac{\tau^x}{\rho_0 \beta L H}, \quad c = \sqrt{g'H} \tag{24}$$

The boundary condtions are given by $v|_{y=0} = 0$ and $v|_{y\to\infty} = 0$.

c.

make (23) dimensionless by letting $y = L\hat{y}$, $(u, v) = a(\hat{u}, \hat{v})$, and $t = (\beta L)^{-1}\hat{t}$. Use a tridigaonal, for instance Gauss eliminatio, to solve the dimensionless expression of (23). Let $\Delta y = 0.1$ and plot $\hat{v}$ and $\hat{u}$ at time $\hat{t} = 1$ as a function of $\hat{y}$ from $\hat{y} = 0$ to $\hat{y} = 8$. Note that $\hat{v}$ is different from zero at $\hat{y} = 8$. You have to figure out for yourself how to obtain the condition at $\hat{y} \to \infty$.

d.

Discuss briefly the solution. Let $\tau^x = 0.1Pa$, $\beta = 2.\cdot 10^{-11}(ms)^{-1}$, $L = 275$km, $\rho = 10^3 kgm^{-3}$ and $H = 200m$. What is the maximum current in the equatorial jet for $\hat{t} = 1$?

e.

Solve (23) analytically. Hint: Make a series using Hermitian polynomials (se for instance *Abramowitz and Stegun*, 1965).

Computer problem 5:
Multiple solutions - Rossby and gravity waves

We will solve the one dimensional shallow water equations to demonstrate multiple solution modes, the role of initial conditions and the use of the Flow Relaxation Scheme (FRS) as an open boundary condition.

We assume that the derivatives with respect to the y-direction are zero, except for the background pressure force which gives rise to a (constant) geostrophic flow, say u_g, in the x-direction. Let u, v be the the velocity components along, respectively, the x, y-axes, and h the geopotential height. Then the one dimensional, shallow water equations are are

$$\partial_t h = -u\partial_x h - h\partial_x u, \tag{25}$$

$$\partial_t u = fv - u\partial_x u - g\partial_x h, \tag{26}$$

$$\partial_t v = -fu - u\partial_x v + fu_g, \tag{27}$$

where f is the Coriolis parameter that we will assume is constant and given by $f = 1.26 \cdot 10^{-4}$s^{-1} which is its value at 60°N.

Initially the fluid is at rest, i.e., $u = v = 0$, while the geopotential height is given by

$$h = H + Ae^{-\left(\frac{x-x_m}{x_0}\right)^2}, \tag{28}$$

where $H = 1000$m, $A = 15$m, $u_g = 0$ms^{-1} and x_m is the middle point of the domain.

a)

Show that by introducing $U = hu, V = hv$ and h as the new variables the above equation becomes

$$\partial_t h = -\partial_x U, \tag{29}$$

$$\partial_t U = fV - \partial_x \left(\frac{U^2}{h}\right) - \frac{1}{2} g\partial_x h^2, \tag{30}$$

$$\partial_t V = -f(U - U_g) - \partial_x \left(\frac{UV}{h}\right). \tag{31}$$

where $U_g = hu_g$.

b)

To solve the above equations numerically we will adopt the leapfrog scheme. Show that the numerical algorithm then becomes

$$h_j^{n+1} = h_j^{n-1} - \frac{\Delta t}{\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right), \tag{32}$$

$$U_j^{n+1} = U_j^{n-1} + 2fV_j^n \Delta t + ADVU + PRES, \tag{33}$$

$$V_j^{n+1} = V_j^{n-1} - 2f(U_j^n - U_g)\Delta t + ADVV \tag{34}$$

where

$$ADVU = -\frac{\Delta t}{\Delta x}\left(\left[\frac{U^2}{h}\right]_{j+1}^n - \left[\frac{U^2}{h}\right]_{j-1}^n\right), \tag{35}$$

$$ADVV = -\frac{\Delta t}{\Delta x}\left(\left[\frac{UV}{h}\right]_{j+1}^n - \left[\frac{UV}{h}\right]_{j-1}^n\right) \tag{36}$$

constitute the advection terms and

$$PRES = -\frac{g\Delta t}{2\Delta x}\left(\left[h^2\right]_{j+1}^n - \left[h^2\right]_{j-1}^n\right) \tag{37}$$

is the pressure term.

c)

Solve the above equations for the domain $x \in \langle 0, D\rangle$ using the above scheme. Assume that the variables u, v, h retain their initial values at the boundaries $x = 0$ and $x = D$. Further, let the grid length be $\Delta x = 100$km, $D = 32\Delta x$ and $x_0 = 5\Delta x$.

How long time step Δt can be used? Explain your choice[1].

As is common we may regard h as the geopotential height of a pressure surface in the atmosphere and as the depth of a water column in the ocean. H is then equilibrium height in the atmosphere associated with a pressure surface of ≈ 900hPa, while it is the equilibrium depth in the ocean. In the latter case $h - H$ is the deviation of the surface away from its equilibrium position.

Plot h or $h - H$ after $1.5, 3.0, 4.5$, and 6 hours into the future. Discuss the solution.

d)

Repeat the above experiment using the the FRS method to relax the inner solution $\tilde{u}, \tilde{v}, \tilde{h}$ towards the externally specified values $(\hat{u}, \hat{v}, \hat{h}) = (0, 0, H)$ in a buffer zone where the relaxation parameter λ_j is given by $1.0, 0.69, 0.44, 0.25, 0.11, 0.03$, and 0, where the first value is used at $j = 1$ and $j = 33$, the next value for $j = 2$ and $j = 32$, etc.

Discuss by comparing this solution to the solution of experiment **c**). Then compute the geostrophic component of the velocity

$$v_g = \frac{g}{f}\partial_x h \tag{38}$$

after $t = 6$ hours, that is, at the end of the integration. Compare v_g and v. What do you think have happened?

[1] Hint: Neglect the non-linear terms and then proceed using von Neumanns method.

f)

Note: This part is not obligatory

Repeat the experiment of **c)** replacing the initial condion by

$$h = H + Ae^{-\left(\frac{x-x_m}{x_0}\right)^2}, \tag{39}$$

$$u = 0, \tag{40}$$

$$v = \frac{g}{f}\partial_x h. \tag{41}$$

Discuss the solution by comparing it to that of experiment **c)**.

Computer Problem 7:
The storm surge problem

In this problem we will consider the so called storm surge problem. This gives you an experience in constructing numerical solutions to problems that includes more than one dependent variable.

In contrast to the atmosphere the astronomical forcing gives rise to an important periodic water level response called tides. In addition to this phenomenon the water level in the ocean also changes due to atmospheric wind and sea level pressure. The latter is called the storm surge response and the water level change caused by it the storm surge. From time to time the joint occurence of high tides and high storm surges can lead to devastating high water levels even along the Norwegian coast. One such example is from mid October 1987 where the water level in Oslo Harbour reached 1.96 meters above normal sea level. In fact since the early 1980s the Norwegian Meteorological Institute has forecasted sea level changes due to storm surges using numerical models.

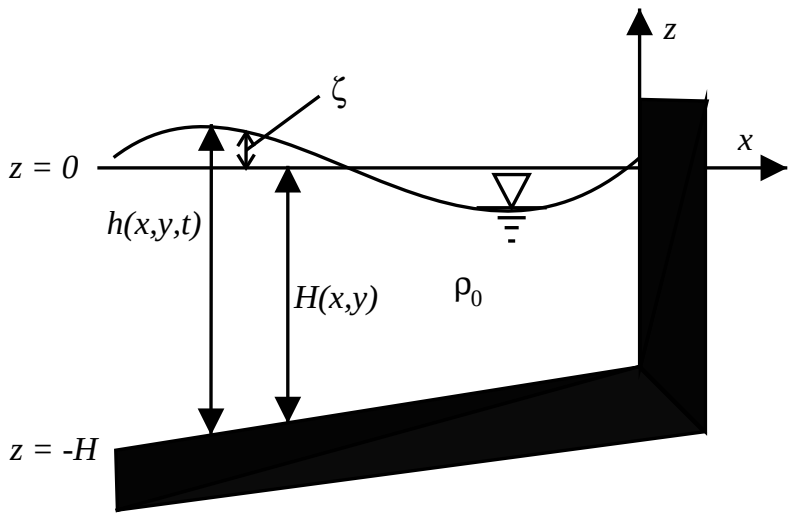

Figure 3: Sketch of a storm surge model along a straight coast conveniently showing some of the notation used.

Many of the earlier studies of storm surges, (e.g., *Røed*, 1979; *Gjevik and Røed*, 1976; *Martinsen et al.*, 1979, to mention a few of the Norwegian ones), have shown that the storm surge is mainly a barotropic response. Storm surge models therefore commonly assume that the density is constant in time and space. The equations therefore reduce to the well known shallow water eqautions. Let

$$\mathbf{U} = \int_{-H}^{\zeta} \mathbf{u} dz, \tag{42}$$

with components (U, V) along the x, y-axes, respectively, be the transport of water in a water column of depth $h = H + \zeta$ where ζ is the sea level deviation away from the equilibrium depth

H (see Figure 3). Then the shallow water equations may be written

$$\begin{aligned} \partial_t \mathbf{U} + \nabla_H \cdot (h^{-1}\mathbf{U}\mathbf{U}) + f\mathbf{k} \times \mathbf{U} = -gH\nabla_H(h-H) + \rho_0^{-1}(\boldsymbol{\tau}_s - \boldsymbol{\tau}_b), \\ \partial_t h + \nabla_H \cdot \mathbf{U} = 0. \end{aligned} \tag{43}$$

where $\boldsymbol{\tau}_s$ and $\boldsymbol{\tau}_b$ are respectively the wind and bottom stresses with components (τ_s^x, τ_s^y) og (τ_b^x, τ_b^y), g is the gravitational acceleration and ρ_0 is the (uniform in time and space) density. Linearizing (43) and neglecting variations in the y direction then gives

$$\begin{aligned} \partial_t U - fV = -gH\partial_x h + \rho_0^{-1}(\tau_s^x - \tau_b^x), \\ \partial_t V + fU = \rho_0^{-1}(\tau_s^y - \tau_b^y), \\ \partial_t h + \partial_x U = 0. \end{aligned} \tag{44}$$

In the following we will assume that changes in the equilibrium depth are so small that H to a good approximation can be considered as being constant.

a.

Show that (44) follows by linearizing (43) under the assumption that changes in the equilibrium depth H are insignificant and that $|\mathbf{U}|^2 \ll |\mathbf{U}|$.

b.

What changes are introduced to (44) if the changes in the equilibrium depth H are significant?

c.

We will solve (44) using numerical methods. To this end we will use a centered in space and forward-backward in time scheme[2]. Hence, one such scheme, called the Sielecki scheme (*Sielecki*, 1968), is

$$\begin{aligned} \frac{U_j^{n+1} - U_j^n}{\Delta t} &= fV_j^n - gH\frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} + \frac{(\tau_s^x)_j^n - (\tau_b^x)_j^n}{\rho_0}, \\ \frac{V_j^{n+1} - V_j^n}{\Delta t} &= -fU_j^{n+1} + \frac{(\tau_s^y)_j^n - (\tau_b^y)_j^n}{\rho_0}, \\ \frac{h_j^{n+1} - h_j^n}{\Delta t} &= -\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x}. \end{aligned} \tag{45}$$

Here we have assumed that all variables are evaluated at the same point in time and space, that is, are evaluated in a non-staggered grid (Arakawa A-grid). Show that the scheme (45) is

[2]Forward-backward in time means that as soon as one dependent variable is updated (in time) we use these values when updating the other dependent variables.

numerically stable under the condition

$$\Delta t \leq \frac{2\Delta x}{\sqrt{gH}}\sqrt{1-\left(\frac{f\Delta t}{2}\right)^2} \tag{46}$$

using von Neumanns method[3].

In the following we will first solve the storm surge problem by hand. Although the analytic solution only constitutes an approximation to the problem it can nevertheless be used to verify that the numerical solution is well behaved. ng analytic methods det videre skal vi se på løsninger av stormfloproblemet langs en rett kyst, that is, analytiske og numeriske løsninger av det lineariserte systemet (44). For dette formål skal modellens parametere, dersom intet annet er nevnt settes som følger,

$$\begin{aligned} \boldsymbol{\tau}_s &= (0, 0.1)\text{Pa} \quad , \qquad \rho_0 = 10^3 kg/m^3, \\ \boldsymbol{\tau}_b &= \rho_0 R \frac{\mathbf{U}}{H} \quad , \qquad R = 2.4 \cdot 10^{-3} m/s. \end{aligned} \tag{47}$$

Begynnelsestilstanden er gitt ved at havet ved tiden er i ro og i likevekt, that is,

$$\mathbf{U}(x,0) = 0 \quad \text{og} \quad \zeta = 0. \tag{48}$$

Merk at havet bare er begrenset av den rette kysten ved $x = 0$. Her forlanges at det ikke strømmer vann gjennom kysten altså at $\mathbf{i} \cdot \mathbf{U} = 0$. For $x \to -\infty$ har vi en åpen rand. Her er det rimelig å forlange at løsningen tilnærmet er lik Ekmanløsningen, that is, her er det ingen endringer i havets vannstand,

$$h|_{x\to-\infty} = H \text{ eller } \partial_x h|_{x\to-\infty} = 0. \tag{49}$$

d.

Vis at svingninger på treghetsfrekvensen sløyfes dersom vi ser bort ifra leddet $\partial_t U$ i (44). Sannsynliggjør også at Ekmanløsningen (49) er den naturlig randbetingelsen når $x \to \infty$.

e.

Anta geostrofisk balanse i y-retningen (that is hastigheten langs kysten balanseres av trykkkraften på tvers av kysten) som matematisk reduserer første likning i (44) til

$$fV = gH\partial_x h. \tag{50}$$

[3]Hint: When analysing the instability neglect all forcing (stress) terms. Also let the discrete Fourier representation of the dependent variables be $U = U_n e^{i\alpha(j-1)\Delta x}$, $V = V_n e^{i\alpha(j-1)\Delta x}$, and $h = H_n e^{i\alpha(j-1)\Delta x}$, respectively. To arrive at (46) eliminate first U_n, U_{n+1} and V_n, V_{n+1} to arrive at one equation involving only H_n and H_{n+1}.

Vis så at den analytiske løsningen, under forutsetning av at (47) gjelder med $R = 0$ (ingen bunnspenning), kan skrives på formen

$$U = U_E \left(1 - e^{x/\lambda}\right) \tag{51}$$

$$V = ftU_E e^{x/\lambda} \tag{52}$$

$$h = H\left(1 + \frac{tU_E}{\lambda H} e^{x/\lambda}\right) \tag{53}$$

hvor $\lambda = \sqrt{gH}/f$ er Rossbys deformasjonsradius og

$$U_E = \frac{\tau_s^y}{\rho_0 h}, \tag{54}$$

er Ekmanhastigheten, that is, den hastigheten du får ved å løse den stasjonære utgaven (that is $\partial_t = 0$) av (44) med $\partial_x = 0$ og $R = 0$.

f.

Søk deretter en analytisk løsning hvor vi tar med bunnspenningen i y-retningen, that is, $\tau_b^x = 0, \tau_b^y = \rho_0 Rv$. Se også her bort ifra treghetssvingningene ved å anta geostrofisk balanse i y-retningen. (Hint: Gjør bruk av Laplace transformasjoner).

g.

Fremstill løsningene av h, U og V under punktene e. og f. grafisk i hvert sitt $x - t$ diagram (Hovmøller diagram).

h.

Løs deretter stormfloproblemet numerisk med bruk av fulle bunnspenninger. Bruk et forskjøvet gitter slik at h-punktet er forskjøvet i forhold til U, V-punktene og slik at h-punktene ligger midt mellom U, V-punktene. Avstanden mellom h-punktene er Dx. Velg denne slik at Rossbys deformasjonsradius, λ, er oppløst, that is, $\Delta x \simeq \lambda/10$. Anta videre at Ekmanløsningen er gyldig langt fra kysten, that is, i en avstand mye lengre enn Rossbyradien, that is minst en størrelsesorden ($x = 10\lambda$), og bruk dette som randkrav for $x >> \lambda$. Randkravet for $x = 0$ er som angitt en "slip" betingelse, that is $U = 0, V \neq 0$. Fremstill løsningen av h, U og V grafisk i et Hovmøller diagram, og sammenlign med de analytiske løsningene. Drøft eventuelle forskjeller og likheter.

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